Ramsey-Cass-Koopmans Model: A Numerical Method Analysis

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Abstract

The Ramsey-Cass-Koopmans model, or Ramsey growth model, is a neoclassical model of economic growth which is one of the most popular and widely used macroeconomic models. The dynamics of the economy in this model is defined by dynamics of capital and consumption which is captured by system of ordinary differential equation for capital and consumption. In this project, we tried to approximate the dynamics of capital and consumption using fourth order Runge-Kutta method.

1 Introduction

First introduced by Ramsey in 1928, a sophisticated model of a societys optimal saving, this early model of Ramsey had to pass three decades until his contribution was taken up seriously (Samuelson and Solow, 1956). The version of the model which we present below was completed by the work of Cass (1965) and Koopmans (1965). Hence the model is also known as the Ramsey-Cass-Koopmans model.

In the Ramsey model there is a finite number of agents with an infinite time horizon; further, these agents are completely alike. The Ramsey model is thus a representative agent model. Originally Ramsey set out the model as a central planner's problem of maximizing levels of consumption over successive generations.Only later was a model adopted by Cass and Koopmans as a description of a decentralized dynamic economy. The RamseyCassKoopmans model aims only at explaining long-run economic growth rather than business cycle fluctuations, and does not include any sources of disturbances like market imperfections, heterogeneity among households, or exogenous shocks. Subsequent researchers therefore extended the model, allowing for government-purchases shocks, variations in employment, and other sources of disturbances, which is known as real business cycle theory.

Our emphasis is to apply numerical method to analyze the dynamics of consumption and capital accumulation under RCK model where dynamics of the economy extends successive generations over time. In this pursuit, the classical Runge-Kutta numerical method is applied to the Ramsey model's ordinary differential equations of capital accumulation and consumption. The Runge-Kutta method confirms the fourth order of convergence after checking with differential equations of capital and consumption of RCK model. For various initial values of capital and consumption, the Runge-Kutta methods provides different dynamic path under this model where some path lead to equilibrium saddle path and some path diverge from saddle path based on the magnitude of initial capital and consumption.

2 The RCK Model

The Ramsey-Cass-Koopmans model, hereafter just RCK model, starts with an aggregate production function that satisfies the Inada conditions, often specified to be of Cobb-Douglas type $F(K, L)$ with factors capital (K) and labor (L). Since this production function is assumed to be homogeneous of degree 1, one can express it in per capita terms. Since technology is also an important factor, we included technology (A)

in the production function. So, our production function is

$$Y = F(K, AL)$$

where Y is output or production,

K is Capital with motion function $\dot{K} = Y - C - \delta K$,

A is Technology/Knowledge with motion function $\dot{A} = gA$,

L is Labor with motion function $\dot{L} = nL$

Here, C is consumption, δ is depreciation rate of capital, g is technology or knowledge growth rate and n is labor growth rate.

We can express this production function in per capita effective labor form which is $y = f(k)$ where $y = Y/AL$, $k = K/AL$. The first key equation of the RCK model is the state equation for capital accumulation:

$$\dot{k} = f(k) - (n + g + \delta)k - c$$

a non-linear differential equation,

The second equation of the model is the solution to the social planner's problem of maximizing a social welfare function,

$$U_0 = \int_0^{\infty} e^{-(\rho - n)t} u(c) dt$$

where $\rho > 0$ is a discount rate reflecting time preference.
The social welfare function consists of the stream of exponentially discounted instantaneous utility from consumption,

$$u(c) = \frac{c^{1-\theta}}{1-\theta}, \qquad \theta > 0$$

Social planner must optimize utility given the constraint which is capital motion ($\dot{k}$). So, social planner's optimization problem will be

$$\max_c U_0 = \int_0^{\infty} e^{-(\rho - n)t} u(c) dt$$

$$st. \qquad c = f(k) - (n + \delta)k - \dot{k}$$

Solving this problem, for instance by converting it into a Hamiltonian function, yields a non-linear differential equation that describes the optimal evolution of consumption,

$$\dot{c} = \frac{1}{\theta} \left[f'(k) - \delta - \rho - \theta g \right] c$$

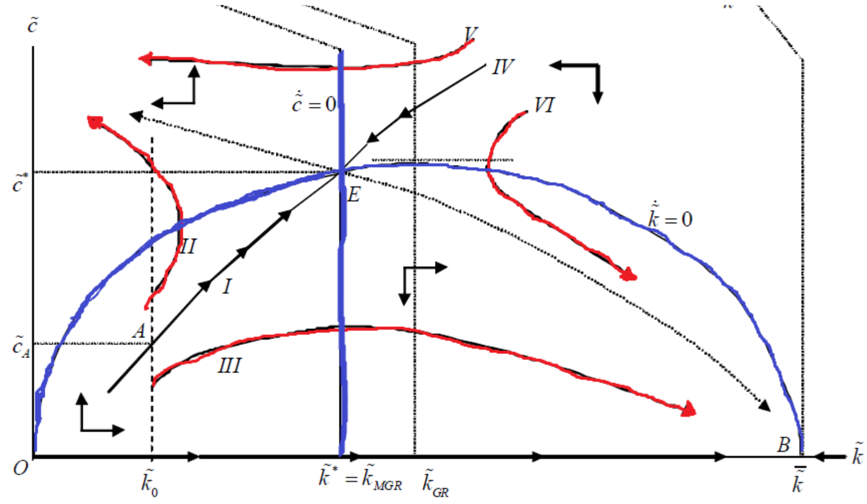

Figure 1: Phase diagram of RCK model

Together both differential equations describe the RCK model's dynamic system. The steady state, which is found by setting $\dot{k}$ and $\dot{c}$ equal to zero, is given by the pair (k^*, c^*)defined by

$$f'(k^*) = \delta + \rho + \theta g \quad and \quad c^* = f(k^*) - (\delta + g + n)k^*$$

The dynamism in this model economy, defined by the above to system of differential equation, can be viewed in the following phase diagram -

3 The IVP and RK Method

The goal of the initial value problem (IVP) is to find a function $y(t)$ given its value at some initial time t_0 and a first order differential equation $f(t, y)$:

$$\begin{cases} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

In applications we want to plot an approximation to $y(t)$ over a designated interval of interest $[t_0, t_{max}]$ in an effort to discover qualitative properties of the solution.
One of the most common numerical techniques used to solve initial value problems is the fourth-order Runge-Kutta technique. For step size $h > 0$, the general form of fourth order RK method is

$$r_1 = f(t_i, y_i)$$

$$r_2 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2} r_1)$$

$$r_3 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2} r_2)$$

$$r_4 = f(t_i + h, y_i + h r_3)$$

$$y_{i+1} = y_i + \frac{h}{6}[r_1 + 2r_2 + 2r_3 + r_4]$$

Considering RCK model in last section, we can develop the IVP as follows -

$$\begin{cases} dF = f(t,k,c) = \Big[f(k) - (n+g+\delta)k - c, \quad \frac{1}{\theta}\Big(f'(k) - \delta - \rho - \theta g\Big)\Big] \\ F(0) = [k(0) \quad c(0)] \end{cases}$$

for time interval $[t_0, T]$.

Here, $f(k) = k^\alpha, \quad \alpha > 0$ is production function,

n is population growth rate,

g is technology growth rate,

δ is depreciation rate,

θ is intertemporal elasticity of substitution,

ρ is discount rate of time preference.

In the next section, we will try to approximate the solution of this IVP of RCK model, which will ultimately results in phase diagram, by using the Runge-Kutta method.

4 Numerical Method Analysis

4.1 Efficiency of RK method

Before using the fourth order RK method to approximate RCK model IVP, we will check the efficiency of the method to approximate any IVP.

Consider the following IVP: $\begin{cases} f(t,k) = -k \\ k(0) = 1 \end{cases}$ for interval $[0, \ 1]$

We know the exact solution of this IVP is $y(t) = e^{-t}$. Now, we will apply the fourth order RK method to approximate the given IVP and compare the result with exact solution.

Here we calculated the maximal absolute errors for each h following the formula $\max_{0 \le i \le N} |e_i|$ where e_i is the difference of approximate y and true y for each $i = 1, 2, \cdots, N$.

Maximal errors for each H are

$$
\begin{aligned}
\text{for } h = 0.2, \quad e_1 &= \max_{0\le i\le N} |e_i| = 0.5797 \times (1/100000),\\
\text{for } h = 0.1, \quad e_2 &= \max_{0\le i\le N} |e_i| = 0.0333 \times (1/100000),\\
\text{for } h = 0.05, \quad e_3 &= \max_{0\le i\le N} |e_i| = 0.0020 \times (1/100000),\\
\text{for } h = 0.025, \quad e_4 &= \max_{0\le i\le N} |e_i| = 0.0001 \times (1/100000).
\end{aligned}
$$

Here the errors are very very small. We can see as the step size h_j decreases, the maximal absolute error is decreased which implies our approximation gets better with decrease in step size.

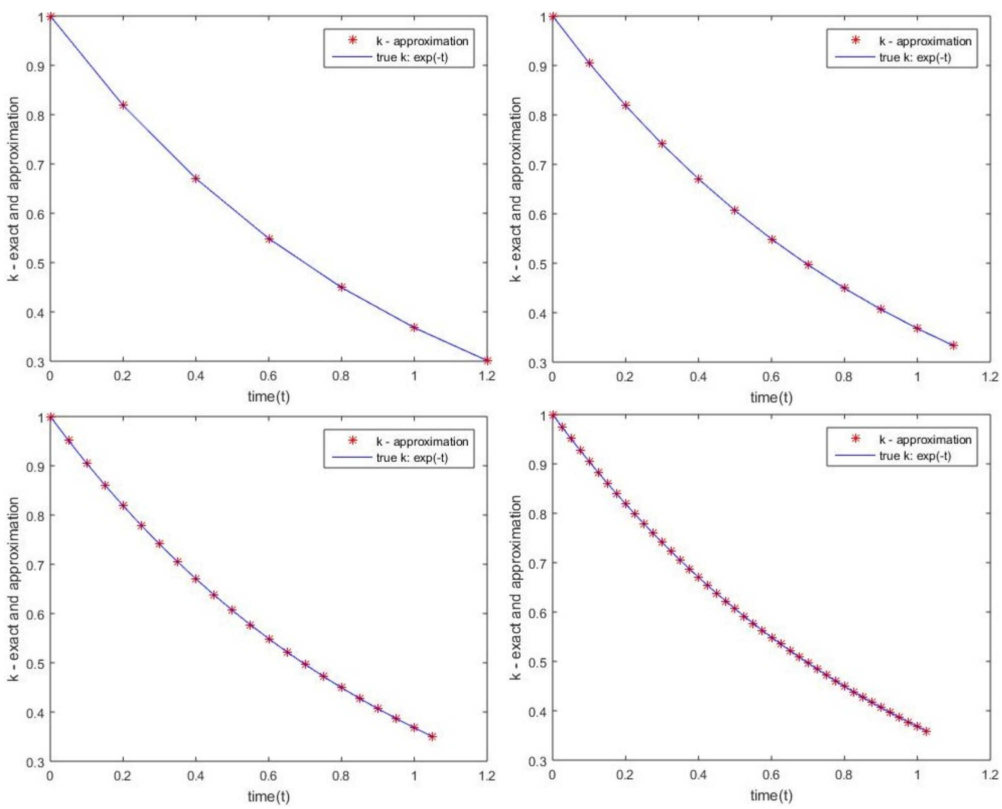

Figure 2: Approximation by RK method for different step size h

4.2 Order of Convergence

This is the classical Runge-Kutta method and y_{i+1} is the fourth RK approximation of $y(t_{n+1})$. Since we know that if $\frac{e_1}{e_2} = \frac{e_2}{e_3} = \cdots = \frac{e_n}{e_{n+1}} = 2^p$, then the order of the method is p. We can do numerical testing of order. In this problem, we can see that, $h_2 = \frac{h_1}{2}$, $h_3 = \frac{h_2}{2}$ and $h_4 = \frac{h_3}{2}$. Also observe that $\frac{e_1}{e_2} \approx 17.39$, $\frac{e_2}{e_3} \approx 16.681$ and $\frac{e_3}{e_4} \approx 16.337$. This shows that it as we decrease the step size ($h \to 0$), the ratio (e_h/e_{h+1}) goes to 16.

$$\lim_{h\to 0} \frac{e_h}{e_{h+1}} = 16 = (2)^4.$$

Here, $\frac{e(h)}{e(h+1)} \approx (2)^4$. Hence, order $p = 4$.

4.3 Approximation of dynamics of RCK Model

In the last section we found that Runge-Kutta numerical method is good enough to approximate IVP, hence we will use it to approximate the IVP of RCK model. The dynamics of the economy under RCK model converge to the saddle path and diverge from saddle path based on the magnitude of different initial values of capital and consumption and also different parameters value.

After a lot trial and error, we have assumed various parameters value and also the initial values for capital accumulation and consumption to approximate the RCK model. The parameters are -

$\delta = 0.1$ which is capital depreciation rate,

$g = 0.05$ which is technology growth rate,

$n = 0.01$ which is population growth rate,

$\alpha = 0.5$ which is production function exponent,

$\rho = .1$ which is rate at which consumption is discounted, and

$\theta = 1$ which is intertemporal elasticity of Substitution.

Given the parameter values, the equilibrium capital and consumption of this dynamic economy will be -

$$k^* = \left(\frac{\alpha}{\delta + \rho + \theta g}\right)^{\frac{1}{1-\alpha}} = 4$$

$$c^* = (k^*)^\alpha - (\delta + g + n)k^* = 1.36$$

Applying the Runge-Kutta method and taking the given parameter values, we tried numerous initial points of capital (K) and consumption (C). The following four plots (figure 3) show some possible trajectories of the economy from different initial points. The equilibrium point is illustrated by the intersection of two perpendicular blue line on horizontal and vertical axes.

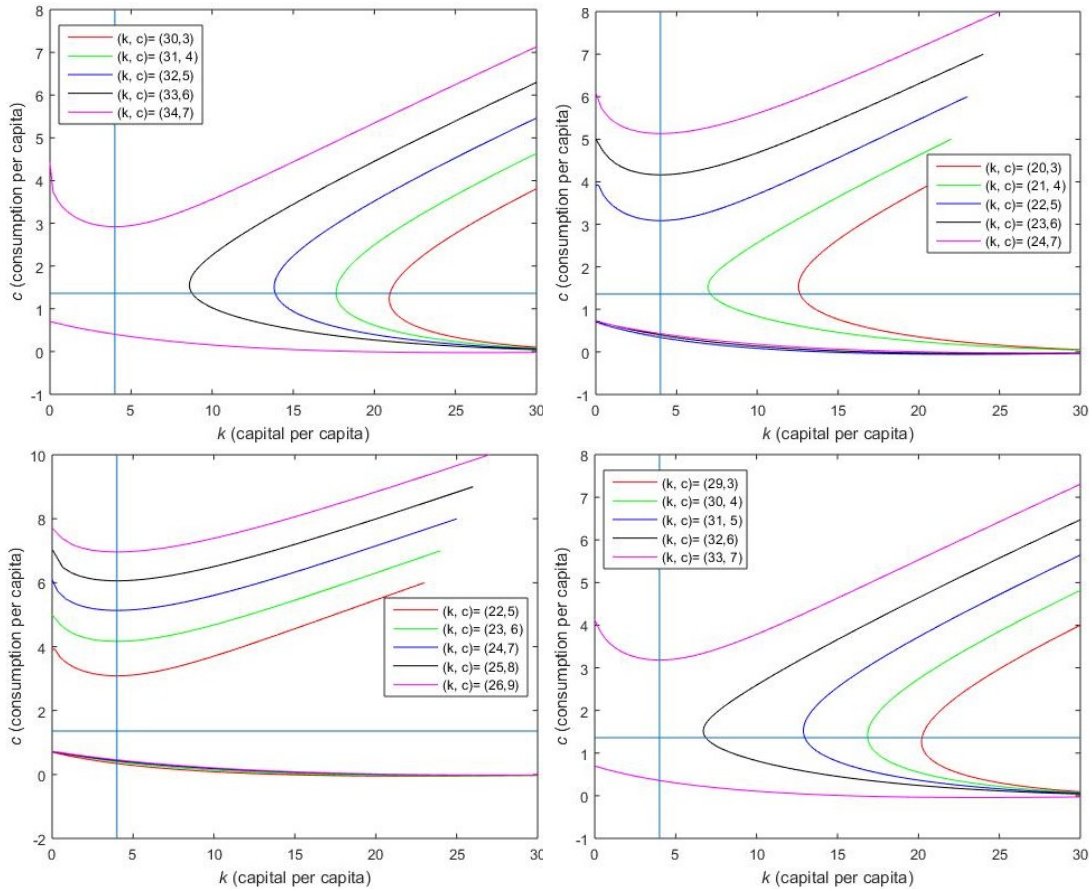

Figure 3: Phase Diagram of RCK Model with different Initial values

The RCK model also explains the dynamics between capital and consumption with the illustration of phase diagram. The economy converges to stable equilibrium from different starting point of initial K and C from the saddle path. The saddle path leads to the stability and satisfies all the constraint in the model. Small deviation from the saddle path leads the economy to the dynamic path. This can be seen in the above plot 4 (figure 3). The trajectory staring from $(k(0) = 32, c(0) = 6)$ goes to totally different way than the trajectory starting from $(k(0) = 33, c(0) = 7)$. A small change in the initial point causes drastic change in the trajectory.

5 Conclusion

RCK model is one of the most used macroeconomic model in economics. In this project we tried to approximate the dynamics of the RCK economy by numerical method. For pursuing that we used fourth order Runge]-Kutta method and we also showed that this RK method is convergent and has order 4. This assures that the approximation of dynamics RCK is quite reliable.

By RK method, We have shown that from different initial condition of capital and consumption economy can move to different trajectories. Further study can be done by showing how the phase diagram (trajectories of capital and consumption) behaves for change in different parameters.