Spring Pendulum System

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Abstract

This project uses an Explicit Runge Kutta Method to approximate the solution of the first order initial value problem formed from the equations of motion of a spring pendulum system.Through, Matlab an animation of the spring pendulum is created and it captures the motion of the spring pendulum given some initial conditions. From which the motion of the spring pendulum as a function of angle of displacement (θ) is studied.We find out how θ affects the stretch in spring as well as the amplitude of pendulum and at what angle we get the maximum extension in spring and amplitude of pendulum, when the spring is initially not stretched. We also find out what types of motion come from different stretches of the spring in the system.

1 Introduction

A spring pendulum (also called elastic pendulum or swinging spring) is a physical system where a piece of mass is connected to a spring so that the resulting motion contains elements of a simple pendulum as well as a spring. The system is much more complex than a simple pendulum, as the properties of the spring add an extra dimension of freedom to the system. For this project, we created a model using Matlab that is able to plot the motion of a spring pendulum system that is set up as in Figure 1. This paper discusses in detail the numerical method used for the project, the initial value problem (IVP) resulting from the equations of motion of the spring pendulum system and the results found.

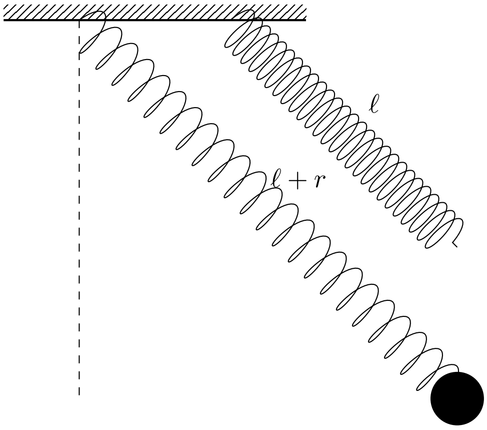


Figure 1: Spring-Pendulum System

2 Numerical Method Description

The Explicit Runge Kutta (ERK) method shown below;

$$\psi_1 = y_n,$$

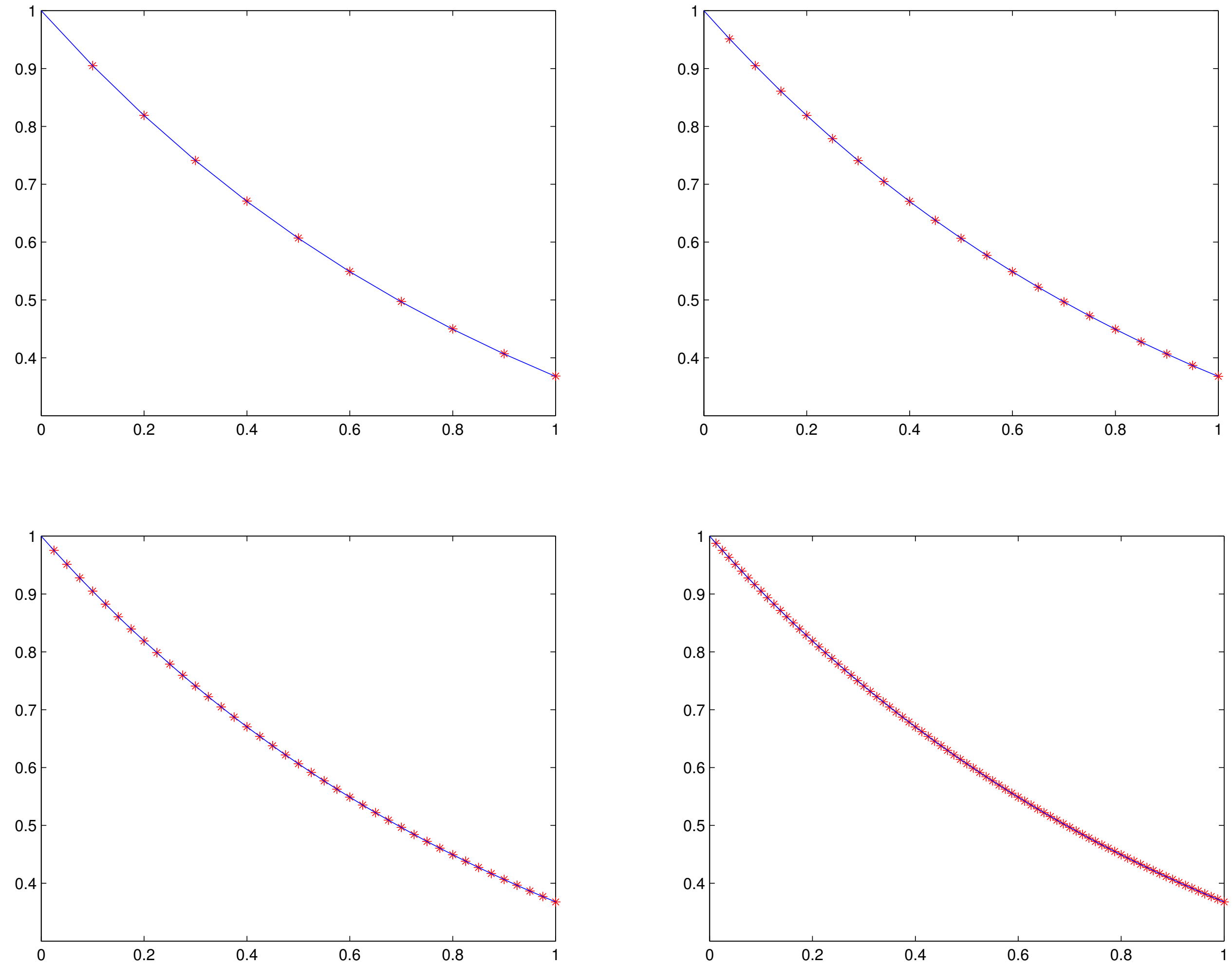
$$\psi_2 = y_n + \frac{h}{2} f(t_n, y_n), \tag{1}$$

$$y_{n+1} = y_n + hf(t_n + \frac{h}{2}, \psi_2) \tag{2}$$

is what is used in approximating the ordinary differential equations in the project. The corresponding tableau is given below

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array}.$$

By writing a Matlab script approximating the solution of $y' = -y$, $y(0) = 1$ over [0,1] with the ERK described above, we were able to find the method's order of convergence. Using $h = .1/2^j$, j=0,1,2,3 and recording maximum absolute the errors as e_j, j=1,...,4 we get $e_1 = 6.6154e-04$, $e_2 = 1.5918e-04$, $e_3 = 3.9049e-05$ and $e_4 = 9.6706e-06$.The ratio of the errors is 4.1559, 4.0764, and 4.0379. The ratio of errors are all approximately equal to $4 = 2^2$.Therefore, we conclude that the order of convergence is 2 and that it is exact for polynomials of degree ≤ 2. Below are the resulting plots of $h = 0.1, 0.05, 0.025$, and 0.0125 respectively (from left to right);



By having $h = t_{n+1} - t_n$ and substituting (1) in (2) the ERK becomes

$$y_{n+1} = y_n + hf(\frac{t_n + t_{n+1}}{2}, y_n + \frac{h}{2} f(t_n, y_n)),$$

which is the explicit-midpoint rule. The explicit-midpoint rule has order 2 and is exact for polynomials of degree ≤ 2, which ascertains our results on the method's order of convergence and exactness results.

A Runge Kutta method solving the IVP

$$y^{'}(t) = \lambda y(t)$$

$$y(t_0) = y_o, \lambda \epsilon \mathbb{C}^-$$

has the form $y_n = (1 + zb^T(I - zA)^{-1}\vec{1})^n y_0$, where $z{=}h\lambda$. The $\lim_{n\to\infty} |y_n| = 0$, if and only if $|1 + zb^T(I - zA)^{-1}\vec{1}| < 1$. Thus, a method is said to be A-stable if and only if $|r(z)| < 1$ for all $z \in \mathbb{C}^-$. For this numerical method;

$$A = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix},$$

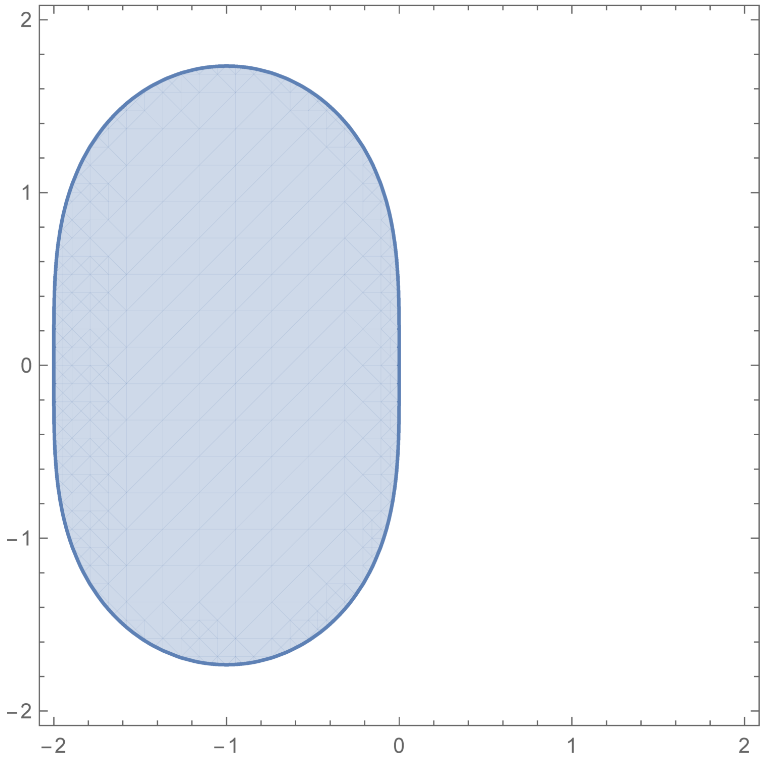
$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$b^T = \begin{bmatrix} 0 & 1 \end{bmatrix}, and$$

$$\vec{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Substituting A, I, b^T and $\vec{1}$ into r(z) we get $r(z) = \frac{z^2}{2} + z + 1$. It is not possible for $\mid \frac{z^2}{2} + z + 1 \mid < 1$ for all $z \in \mathbb{C}^-$, since polynomials are unbounded on infinite sets. The method is therefore not A-stable. We can also see this by plotting $r(z)$ (the method's linear stability domain), using the region plot function in Mathematica. Below is the plot, which also shows the method's linear stability domain is neither a superset nor equal to $\mathbb{C}^-$, and it is therefore the method is not A-stable.

Figure 2: Linear Stability Domain



3 IVP

Consider a mass m attached to a spring of spring constant k swinging in a vertical plane as shown in Figure 1. The equations of motion can be derived easily by writing the Lagrangian and then writing the Lagrange equations of motion, where l is equilibrium length of the pendulum, m is mass of the bob attached to spring, g is acceleration due to gravity measured in m/s^2 and t is time in seconds. Let $l + r(t)$ be the length of the spring, and $\theta(t)$ be the angle of the spring with respect to time as shown in Figure 2.

The equations of motion for spring pendulum are as follows:

$$\frac{d^2r}{d^2t} = (l + r)(\frac{d\theta}{dt})^2 + g\cos\theta - w_r^2 r \tag{3}$$

$$\frac{d^2\theta}{d^2t} = \frac{-2}{l + r}\frac{dr}{dt}\frac{d\theta}{dt} - w_\theta^2 \sin\theta. \tag{4}$$

Where $w_r = \sqrt{\frac{k}{m}}$ is the spring's frequency along its length and $w_\theta = \sqrt{\frac{g}{l+r}}$ is the pendulum's frequency of oscillation. The system of equations (3) and (4) can be time integrated to know the trajectory/ position of the spring pendulum using the ERK method described above. To apply the ERK method, the system of equations are transferred to first order differential equations as follows:
let

$$y_1 = r;$$

$$y_2 = \frac{dr}{dt} = \frac{dy_1}{dt};$$
$$y_3 = \theta;$$
$$y_4 = \frac{d\theta}{dt} = \frac{dy_3}{dt};$$
$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} r \\ \frac{dr}{dt} \\ \theta \\ \frac{d\theta}{dt} \end{bmatrix}$$
$$F(t,y) = \frac{dy}{dt} = \frac{d}{dt}(y) = \frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \tag{5}$$

Since $dy_1/dt{=}y_2$, $dy_3/dt{=}y_4$, our first equation becomes

$$\frac{dy_2}{dt} = (y_1 + l)y_4^2 + g\ cos y_3 - \frac{k}{m}y_1, \tag{6}$$

where

$$w_\theta = \sqrt{\frac{g}{l+r}} = \sqrt{\frac{g}{l+y_1}}$$

and the second equation becomes

$$\frac{dy_4}{dt} = \frac{-2}{l+y_1}y_2y_4 - \frac{g}{l+y_1}\ sin y_3. \tag{7}$$

Using results of (6) and (7) and knowing $dy_1/dt{=}y_2$ and $dy_3/dt\ {=}y_4$ we can rewrite (5) as

$$F(t,y) = \begin{bmatrix} y_2 \\ (y_1 + l)y_4^2 + g\cos y_3 - \frac{k}{m}y_1 \\ y_4 \\ \frac{-2}{l+y_1}y_2y_4 - \frac{g}{l+y_1}\sin y_3 \end{bmatrix}. \tag{8}$$

4 Numerical Results

We created a Matlab script that had an animation of the spring pendulum system using the described ERK method. The initial conditions were g = acceleration due to gravity $= 9.8m/s^2$, l = unstretched spring length $l = 0.5m$,k =spring constant = $20kg$ and m = mass attached to the spring = $2kg$. We begin by showing some different images of the motion of the spring-pendulum using the initial conditions and setting $F'(t,y) = \begin{bmatrix} 0 & 0 & \theta & 0 \end{bmatrix}^T$. When $\theta = 0$ we get the motion shown in Figure 3. There is no y displacement since the spring is only stretched downwards.When $\theta = \frac{\pi}{4}$ the motion is shown in Figure 4 and when $\theta = \frac{\pi}{2}$ the motion is as shown in Figure 5. From the pictures we see that as the angle gets bigger the motion also gets wider.

Figure 3: $\theta = 0$

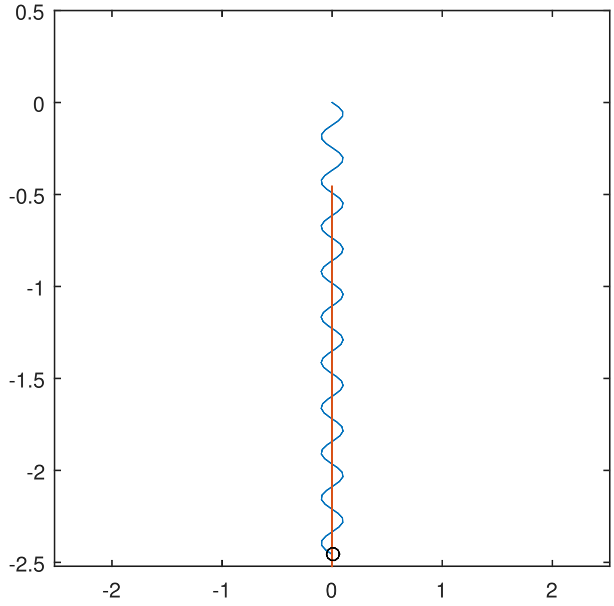


Figure 4: $\theta = \pi/4$

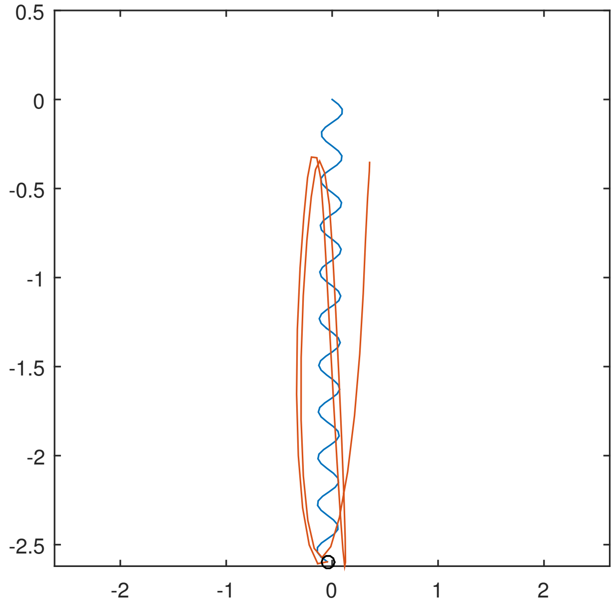
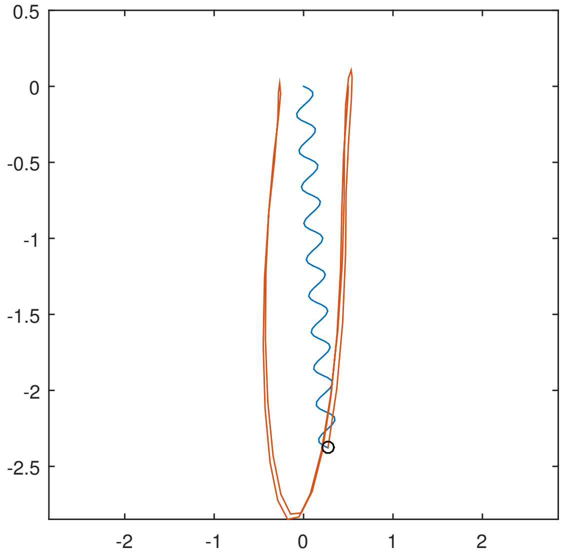


Figure 5: $\theta = \frac{\pi}{2}$



We decided to run the Matlab script with different angles of the form $\frac{n\pi}{16}$, $n = 0, 1..., 8$ and record the maximum x (spring stretch) and maximum y (pendulum amplitude) displacement as shown in Table 1.

Angle θ	Max x displacement	Max y displacement
0	2.5211	0
$\pi/16$	2.5271	0.0975
$2\pi/16$	2.5443	0.1913
$3\pi/16$	2.5705	0.2778
$4\pi/16$	2.6207	0.3536
$5\pi/16$	2.6930	0.4157
$6\pi/16$	2.7954	0.4619
$7\pi/16$	2.8104	0.5726
$8\pi/16$	2.8480	0.5425

Table 1: Angle and Maximum Displacement

In order to find the relationship between the angle and the maximum x and y displacements, we plotted graphs using the values given in Table 1. The graph of the angle (θ) versus the x displacement is shown below by Figure 6. From the graph we see that as the angle increases the x displacement also increases.

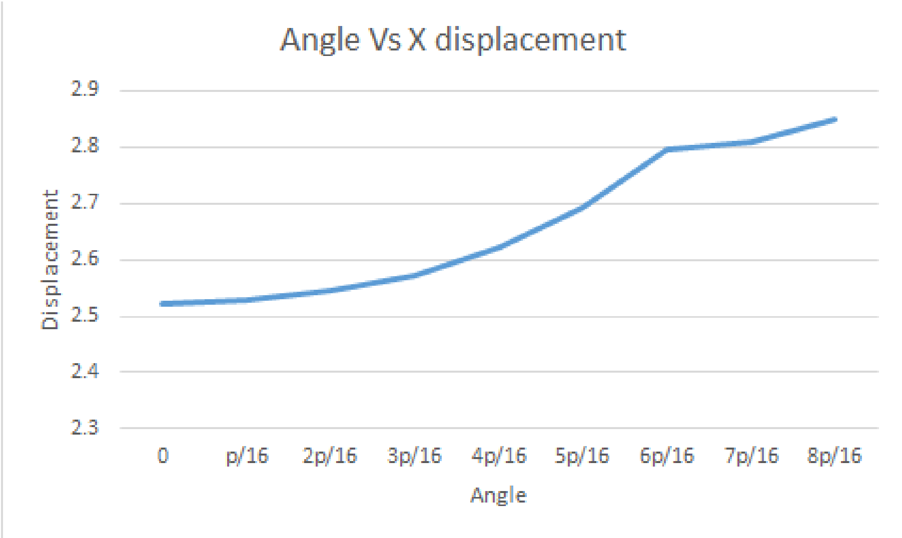


Figure 6: θ versus X Displacement

The graph of the angle (θ) versus y displacement is shown below. From the we see a steady increase in the y displacement from the initial values to when $\theta = \frac{7\pi}{16}$, where there is a peak and then there is a decrease in the displacement value.

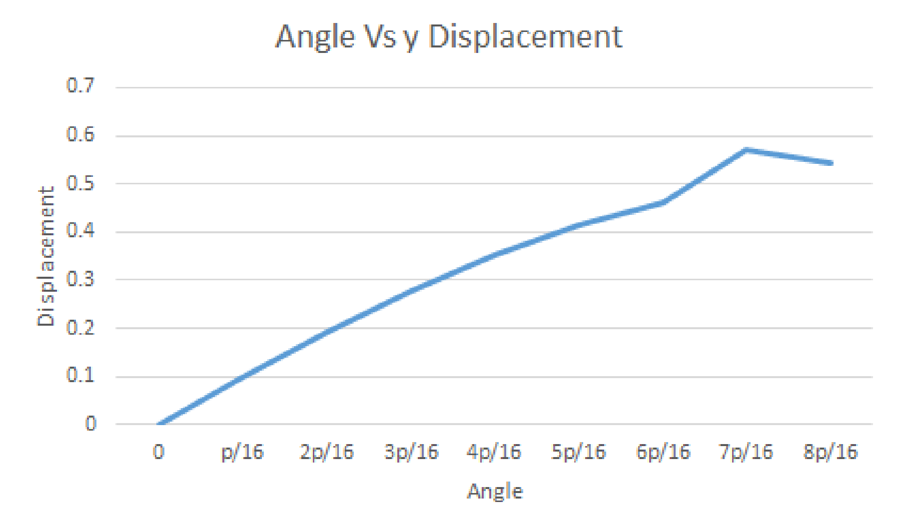


Figure 7: θ versus Y Displacement

We decided to find out what happens when the spring is stretched and see if different patterns of motion will appear. Setting $r=$ the length of the stretch of spring$=0.5$ and having $F'(t, y) = \begin{bmatrix} 0.5 & 0 & \frac{\pi}{4} & 0 \end{bmatrix}^T$, we have the motion shown below in Figure 9.

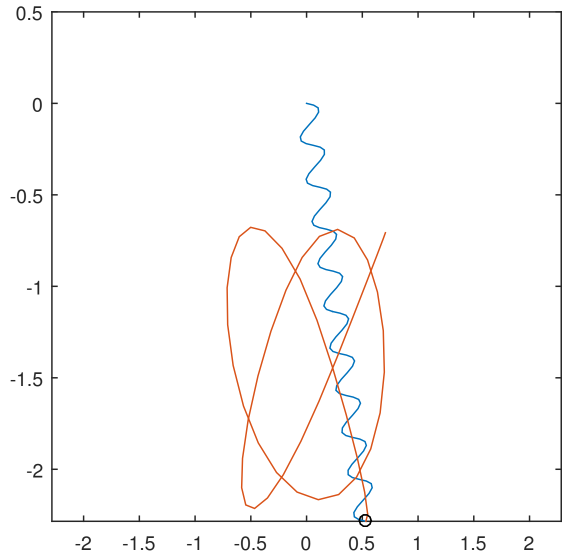


Figure 8: $r = 0.5$

Increasing the stretch by setting $r=$ the length of the stretch of spring$=1$ and having $F'(t,y) = \begin{bmatrix} 1 & 0 & \frac{\pi}{4} & 0 \end{bmatrix}^T$, we have the motion shown below in Figure 8.

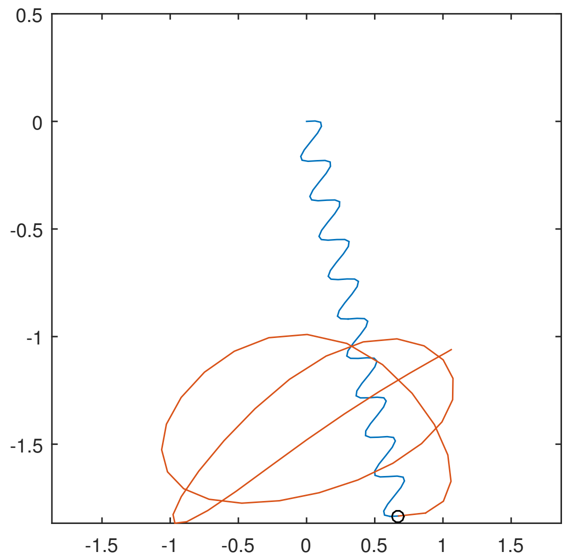


Figure 9: $r = 1$

The motion when the spring is stretched is more circular as compared to when the spring is unstretched. Therefore, depending on the motion of what you would like to model you can either stretch the spring or not.

In conclusion, the spring pendulum system can be used to model real life scenarios. For example, creators of roller coasters can use this project and they can change $F'(t,y)$ to different initial values depending on the impact they would like to achieve. For instance, if they want maximum for both x and y displacement they can increase initial angle, for circular motion they can stretch the spring, and even in choosing the initial velocity the roller coaster they can use the model to find the optimum initial velocity.

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