Problem 1 (20 points): Find the solution of the following linear system of equations

\[
\begin{align*}
  x_1 + x_2 + x_3 &= 4 \\
  2x_1 + 3x_2 - x_3 &= 3 \\
  x_1 + 2x_2 - 3x_3 &= -3
\end{align*}
\]

Solution: We row reduce the augmented matrix corresponding to the system:

\[
\begin{bmatrix}
  1 & 1 & 1 & 4 \\
  2 & 3 & -1 & 3 \\
  1 & 2 & -3 & -3
\end{bmatrix}
\sim
\begin{bmatrix}
  1 & 1 & 1 & 4 \\
  0 & 1 & -3 & -5 \\
  0 & 1 & -4 & -7
\end{bmatrix}
\sim
\begin{bmatrix}
  1 & 1 & 1 & 4 \\
  0 & 1 & -3 & -5 \\
  0 & 0 & 1 & 2
\end{bmatrix}
\sim
\begin{bmatrix}
  1 & 0 & 0 & 1 \\
  0 & 1 & 0 & 1 \\
  0 & 0 & 1 & 2
\end{bmatrix}
\]

The reduced echelon form of the augmented matrix shows that there is a unique solution of the original system, namely:

\[
x_1 = 1, x_2 = 1, x_3 = 2.
\]
Problem 2 (20 points): Write the solution of the following homogeneous system of linear equations in parametric vector form:

\[
\begin{align*}
    x_1 &+ x_2 + x_3 + x_4 = 0 \\
    x_1 + 2x_2 - x_3 + 2x_4 &= 0 \\
    x_2 - 2x_3 + x_4 &= 0
\end{align*}
\]

Solution: We row reduce the augmented matrix of the system to reduced echelon form:

\[
\begin{bmatrix}
    1 & 1 & 1 & 1 & 0 \\
    1 & 2 & -1 & 2 & 0 \\
    0 & 1 & -2 & 1 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
    1 & 1 & 1 & 1 & 0 \\
    0 & 1 & -2 & 1 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
    1 & 0 & 3 & 0 & 0 \\
    0 & 1 & -2 & 1 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The reduced echelon form of the augmented matrix shows that there are two basic variables \(x_1\) and \(x_2\), and two free variables \(x_3\) and \(x_4\). We solve for the basic variables in terms of the free variables:

\[
\begin{align*}
    x_1 + 3x_3 &= 0 \\
    x_2 - 2x_3 + x_4 &= 0 \\
    \text{or} \quad \begin{cases}
        x_1 &= -3x_3 \\
        x_2 &= 2x_3 - x_4
    \end{cases}
\end{align*}
\]

Thus, we obtain the general solution for the system:

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ 2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix},
\]

or, in parametric vector form,

\[
x = s \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \text{ where } s, t \in \mathbb{R} \text{ are parameters.}
\]

2
Problem 3 (20 points): Let

\[
\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}; \text{ and } \mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}.
\]

Is \( \mathbf{b} \) a linear combination of \( \mathbf{a}_1, \mathbf{a}_2 \) and \( \mathbf{a}_3 \)? If so, find weights \( x_1, x_2, x_3 \) such that \( \mathbf{b} = x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + x_3 \mathbf{a}_3 \).

Solution: \( \mathbf{b} \) is a linear combination of \( \mathbf{a}_1, \mathbf{a}_2 \) and \( \mathbf{a}_3 \) if and only if we can find scalars \( x_1, x_2, x_3 \) (weights) such that

\[
x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + x_3 \mathbf{a}_3 = \mathbf{b}.
\]

We row reduce the augmented matrix corresponding to the vector equation above; we get

\[
\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 4 \\ 1 & 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}.
\]

This shows that \( \mathbf{b} \) is indeed a linear combination of \( \mathbf{a}_1, \mathbf{a}_2 \) and \( \mathbf{a}_3 \) with respective weights 2, 1 and 3. That is

\[
\mathbf{b} = 2 \mathbf{a}_1 + \mathbf{a}_2 + 3 \mathbf{a}_3.
\]
Problem 4 (20 points): Show that the three vectors

\[ a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad a_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \]

are linearly independent.

Solution: Recall that \( a_1, a_2 \) and \( a_3 \) are linearly independent if and only if the vector equation

\[ x_1 a_1 + x_2 a_2 + x_3 a_3 = 0 \quad (1) \]

has only the trivial solution. We row reduce the augmented matrix corresponding to the vector equation above; we get

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0
\end{bmatrix}
\sim \cdots \sim
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 2 & 0
\end{bmatrix}.
\]

Note that, since \( a_1, a_2 \) and \( a_3 \) are the same vectors as in the previous problem, obtaining the echelon matrix in this problem is similar to obtaining the echelon form in the previous problem (with \( b = 0 \) in this case.) Inspecting the echelon form of the augmented matrix, we readily see that Equation (1) has only the trivial solution (since there are no free variables); hence \( a_1, a_2 \) and \( a_3 \) are linearly independent.
Problem 5 (20 points): Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be given by

\[
T \left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 - 2x_2 \\ -x_1 + 3x_2 \\ 3x_1 - 2x_2 \end{bmatrix}.
\]

a) Write the standard matrix of \( T \).
b) Does \( T \) map \( \mathbb{R}^2 \) onto \( \mathbb{R}^3 \)? Justify your answer.
c) Is \( T \) one-to-one? Justify your answer.

Solution:

a) The standard matrix is \( A = [T(e_1) \quad T(e_2)] \), where \( e_1 \) and \( e_2 \) are the basic unit vectors of \( \mathbb{R}^2 \). Thus,

\[
T(e_1) = T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \quad \text{and} \quad T(e_2) = T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}.
\]

Hence

\[
A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \\ 3 & -2 \end{bmatrix}.
\]

b) Since \( A \) has more rows than columns (in other words, the codomain of \( T \) has a higher dimension than the domain: \( 3 > 2 \)), \( T \) does not map \( \mathbb{R}^2 \) onto \( \mathbb{R}^3 \). (Why? See notes.)

c) \( T \) is one-to-one if and only if the equation \( T(x) = 0 \) has only the trivial solution, also if and only if the columns of \( A \) are linearly independent (see Theorem 12 in Sec 1.9). But the columns of \( A \) are linearly independent since none of them is a multiple of the other; hence \( T \) is one-to-one.