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Analytical Properties of Power Series on Levi-Civita Fields

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Abstract

A detailed study of power series on the Levi-Civita fields is presented. After reviewing two types of convergence on those fields, including convergence criteria for power series, we study the analytical properties of power series. We show that within their domain of convergence, power series are infinitely often differentiable and re-expandable around any point within the radius of convergence from the origin. Then we study a large class of functions that are given locally by power series and contain all the continuations of real power series. We show that these functions have similar properties as real analytic functions. In particular, they are closed under arithmetic operations and composition; and they satisfy the intermediate value theorem.

1 Introduction

In this paper, a detailed study of the analytical properties of power series on the Levi-Civita fields $\mathcal{R}$ and $\mathcal{C}$ is presented. We recall that the elements of $\mathcal{R}$ and $\mathcal{C}$ are functions from $\mathbb{Q}$ to $\mathbb{R}$ and $\mathbb{C}$, respectively, with left-finite support (denoted by supp). That is, below every rational number q, there are only finitely many points where the given function does not vanish. For the further discussion, it is convenient to introduce the following terminology.

Définition: (λ, $\sim$, $\approx$, $=_r$) For $x \in \mathcal{R}$ or $\mathcal{C}$, we define $\lambda(x) = \min(\text{supp}(x))$ for $x \neq 0$ (which exists because of left-finiteness) and $\lambda(0) = +\infty$.

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Given $x, y \in \mathcal{R}$ or $\mathcal{C}$, we say $x \sim y$ if $\lambda(x) = \lambda(y)$; $x \approx y$ if $\lambda(x) = \lambda(y)$ and $x[\lambda(x)] = y[\lambda(y)]$; and $x =_r y$ if $x[q] = y[q]$ for all $q \leq r$.

At this point, these definitions may feel somewhat arbitrary; but after having introduced an order on $\mathcal{R}$, we will see that λ describes orders of magnitude, the relation $\approx$ corresponds to agreement up to infinitely small relative error, while $\sim$ corresponds to agreement of order of magnitude.

The sets $\mathcal{R}$ and $\mathcal{C}$ are endowed with formal power series multiplication and componentwise addition, which make them into fields [3] in which we can isomorphically embed $\mathbb{R}$ and $\mathbb{C}$ (respectively) as subfields via the map $\Pi : \mathbb{R}, \mathbb{C} \to \mathcal{R}, \mathcal{C}$ defined by

$$\Pi(x)[q] = \begin{cases} x & \text{if } q = 0 \\ 0 & \text{else} \end{cases} . \tag{1.1}$$

Définition: (Order in $\mathcal{R}$) Let $x \neq y$ in $\mathcal{R}$ be given. Then we say $x > y$ if $(x-y)[\lambda(x-y)] > 0$; furthermore, we say $x < y$ if $y > x$.

With this definition of the order relation, $\mathcal{R}$ is a totally ordered field. Moreover, the embedding Π in Equation (1.1) of $\mathbb{R}$ into $\mathcal{R}$ is compatible with the order. The order induces an absolute value on $\mathcal{R}$, from which an absolute value on $\mathcal{C}$ is obtained in the natural way: $|x+iy| = \sqrt{x^2+y^2}$. We also note here that λ, as defined above, is a valuation; moreover, the relation $\sim$ is an equivalence relation, and the set of equivalence classes (the value group) is (isomorphic to) $\mathbb{Q}$.

Besides the usual order relations, some other notations are convenient.

Définition: ($\ll, \gg$) Let $x, y \in \mathcal{R}$ be non-negative. We say x is infinitely smaller than y (and write $x \ll y$) if $nx < y$ for all $n \in \mathbb{N}$; we say x is infinitely larger than y (and write $x \gg y$) if $y \ll x$. If $x \ll 1$, we say x is infinitely small; if $x \gg 1$, we say x is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Non-negative numbers that are neither infinitely small nor infinitely large are also called finite.

Définition: (The Number d) Let d be the element of $\mathcal{R}$ given by $d[1] = 1$ and $d[q] = 0$ for $q \neq 1$.

It is easy to check that $d^q \ll 1$ if and only if $q > 0$. Moreover, for all $x \in \mathcal{R}$ (resp. $\mathcal{C}$), x can be written as $x = \sum_{q \in \text{supp}(x)} \alpha_q d^q$, where $\alpha_q \in \mathbb{R}$

(resp. $\mathbb{C}$) for all $q \in \mathrm{supp}(x)$, and where the series converges in the topology induced by the absolute value [3].

Altogether, it follows that $\mathcal{R}$ and $\mathcal{C}$ are non-Archimedean field extensions of $\mathbb{R}$ and $\mathbb{C}$, respectively. For a detailed study of these fields, we refer the reader to [3, 16, 5, 19, 17, 4, 18]. In particular, it is shown that $\mathcal{R}$ and $\mathcal{C}$ are complete with respect to the topology induced by the absolute value. In the wider context of valuation theory, it is interesting to note that the topology induced by the absolute value, the so-called strong topology, is the same as that introduced in the common way via the valuation λ.

Remark: The mapping $\Lambda : \mathcal{R} \times \mathcal{R} \to \mathbb{R}$, given by $\Lambda(x, y) = \exp(-\lambda(x - y))$, is an ultrametric distance (and hence a metric); the valuation topology it induces is equivalent to the strong topology. Furthermore, a sequence (a_n) is Cauchy with respect to the absolute value if and only if it is Cauchy with respect to the valuation metric Λ.

For if A is an open set in the strong topology and $a \in A$, then there exists $r > 0$ in $\mathcal{R}$ such that, for all $x \in \mathcal{R}$, $|x-a| < r \Rightarrow x \in A$. Let $l = \exp(-\lambda(r))$, then apparently we also have that, for all $x \in \mathcal{R}$, $\Lambda(x, a) < l \Rightarrow x \in A$; and hence A is open with respect to the valuation topology. The other direction of the equivalence of the topologies follows analogously. The statement about Cauchy sequences also follows readily from the definition.

It follows therefore that the fields $\mathcal{R}$ and $\mathcal{C}$ are just special cases of the class of fields discussed in [14]. However, we study in this paper the analytical properties of power series in a topology weaker than the valuation topology used in [14], and thus allow for a much larger class of power series to be included in the study.

Contrary to the real case, the continuity or even the differentiability of a function on a closed interval of $\mathcal{R}$ are not always sufficient for the function to assume all intermediate values, a maximum or a minimum [2, 15]. These deficiencies are not special to $\mathcal{R}$; they are encountered in any totally ordered non-Archimedean field and are due to the total disconnectedness of the field in the topology induced by the order. For a general overview of the algebraic properties of formal power series fields in general, we refer to the comprehensive overview by Ribenboim [13], and for an overview of the related valuation theory the books by Krull [6], Schikhof [14] and Alling [1]. A thorough and complete treatment of ordered structures can also be found in [12].

In addition to the fact that any calculus or analysis on a field remain incomplete without a study of power series, it turns out that for this special class of functions, many of the problems mentioned above can be solved. Previous work on power series on the Levi-Civita fields $\mathcal{R}$ and $\mathcal{C}$ has been mostly restricted to power series with real or complex coefficients. In [8, 9, 10, 7], they could be studied for infinitely small arguments only, while in [3], using the newly introduced weak topology, also finite arguments were possible. Moreover, power series over complete valued fields in general have been studied by Schikhof [14], Alling [1] and others in valuation theory, but always in the valuation topology.

In [19], we study the general case when the coefficients in the power series are Levi-Civita numbers, using the weak convergence of [3]. We derive convergence criteria for power series which allow us to define a radius of convergence η such that the power series converges weakly for all points whose distance from the center is smaller than η by a finite amount and it converges strongly for all points whose distance from the center is infinitely smaller than η.

This paper is a continuation of [19] and complements it: Using the convergence properties of power series on the Levi-Civita fields, discussed in [19], we focus in this paper on studying the analytical properties of power series within their domain of convergence. we show that power series on $\mathcal{R}$ and $\mathcal{C}$ behave similarly to real and complex power series. Specifically, we show that within their radius of convergence, power series are infinitely often differentiable and the derivatives to any order are obtained by differentiating the power series term by term. Also, power series can be re-expanded around any point in their domain of convergence and the radius of convergence of the new series is equal to the difference between the radius of convergence of the original series and the distance between the original and new centers of the series. We then study the class of locally analytic functions and show that they are closed under arithmetic operations and compositions, they are infinitely often differentiable, and they satisfy the intermediate value theorem.

2 Review of Strong Convergence and Weak Convergence

In this section, we review some of the convergence properties of power series that will be needed in the rest of this paper; and we refer the reader to [19] for a more detailed study of convergence on the Levi-Civita fields.

Définition: A sequence (s_n) in $\mathcal{R}$ or $\mathcal{C}$ is called regular if the union of the supports of all members of the sequence is a left-finite subset of $\mathbb{Q}$. (Recall that $A \subset \mathbb{Q}$ is said to be left-finite if for every $q \in \mathbb{Q}$ there are only finitely many elements in A that are smaller than q.)

Définition: We say that a sequence (s_n) converges strongly in $\mathcal{R}$ or $\mathcal{C}$ if it converges with respect to the topology induced by the absolute value.

As we have already mentioned in the introduction, strong convergence is equivalent to convergence in the topology induced by the valuation λ. It is shown that every strongly convergent sequence in $\mathcal{R}$ or $\mathcal{C}$ is regular; moreover, the fields $\mathcal{R}$ and $\mathcal{C}$ are complete with respect to the strong topology [2]. For a detailed study of the properties of strong convergence, we refer the reader to [15, 19].

Since power series with real (complex) coefficients do not converge strongly for any nonzero real (complex) argument, it is advantageous to study a new kind of convergence. We do that by defining a family of semi-norms on $\mathcal{R}$ or $\mathcal{C}$, which induces a topology weaker than the topology induced by the absolute value and called weak topology [3].

Définition: Given $r \in \mathbb{R}$, we define a mapping $\|\cdot\|_r : \mathcal{R}$ or $\mathcal{C} \to \mathbb{R}$ as follows.

$$\|x\|_r = \sup\{|x[q]| : q \in \mathbb{Q} \text{ and } q \leq r\}. \tag{2.2}$$

The supremum in Equation (2.2) is finite and it is even a maximum since, for any r, only finitely many of the $x[q]$'s considered do not vanish.

Définition: A sequence (s_n) in $\mathcal{R}$ (resp. $\mathcal{C}$) is said to be weakly convergent if there exists $s \in \mathcal{R}$ (resp. $\mathcal{C}$), called the weak limit of the sequence (s_n), such that for all $\epsilon > 0$ in $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $\|s_m - s\|_{1/\epsilon} < \epsilon$ for all $m \geq N$.

A detailed study of the properties of weak convergence is found in [3, 15, 19]. Here we will only state without proofs two results which are useful for Sections 3 and 4. For the proof of the first result, we refer the reader to [3]; and the proof of the second one is found in [15, 19].

Theorem 2.1: *(Convergence Criterion for Weak Convergence) Let (s_n) converge weakly in $\mathcal{R}$ (resp. $\mathcal{C}$) to the limit s. Then, the sequence $(s_n[q])$ converges to $s[q]$ in $\mathbb{R}$ (resp. $\mathbb{C}$), for all $q \in \mathbb{Q}$, and the convergence is uniform* *on every subset of* $\mathbb{Q}$ *bounded above. Let on the other hand* (s_n) *be regular, and let the sequence* $(s_n[q])$ *converge in* $\mathbb{R}$ *(resp.* $\mathbb{C}$*) to* $s[q]$ *for all* $q \in \mathbb{Q}$*. Then* (s_n) *converges weakly in* $\mathcal{R}$ *(resp.* $\mathcal{C}$*) to* s*.*

Theorem 2.2: *If the series* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *are regular, if* $\sum_{n=0}^{\infty} a_n$ *converges absolutely weakly to* a *(i.e.* $\sum_{n=0}^{\infty} |a_n - a|$ *converges weakly to* 0*), and if* $\sum_{n=0}^{\infty} b_n$ *converges weakly to* b*, then* $\sum_{n=0}^{\infty} c_n$*, where* $c_n = \sum_{j=0}^{n} a_j b_{n-j}$*, converges weakly to* $a \cdot b$*.*

It is shown [3] that $\mathcal{R}$ and $\mathcal{C}$ are not Cauchy complete with respect to the weak topology and that strong convergence implies weak convergence to the same limit.

3 Power Series

We now discuss a very important class of sequences, namely, the power series. We first study general criteria for power series to converge strongly or weakly. Once their convergence properties are established, they will allow the extension of many important real functions, and they will also provide the key for an exhaustive study of differentiability of all functions that can be represented on a computer [16]. Also based on our knowledge of the convergence properties of power series, we will be able to study in Section 4 a large class of functions which will prove to have similar smoothness properties as real power series. We begin our discussion of power series with an observation [3].

Lemma 3.1: *Let* $M \subset \mathbb{Q}$ *be left-finite. Define*

$$M_{\Sigma} = \{q_1 + \ldots + q_n : n \in \mathbb{N}, \text{ and } q_1, \ldots, q_n \in M\};$$

then M_{Σ} *is left-finite if and only if* $\min(M) \geq 0$*.*

Corollary 3.2: *The sequence* (x^n) *is regular if and only if* $\lambda(x) \geq 0$*.*

Let (a_n) *be a sequence in* $\mathcal{R}$ *(resp.* $\mathcal{C}$*). Then the sequences* $(a_n x^n)$ *and* $(\sum_{j=0}^{n} a_j x^j)$ *are regular if* (a_n) *is regular and* $\lambda(x) \geq 0$*.*

3.1 Convergence Criteria

In this section, we state strong and weak convergence criteria for power series, whose proofs are given in [19]. Also, since strong convergence is equivalent to convergence with respect to the valuation topology, the following theorem is a special case of the result on page 59 of [14].

Theorem 3.3: *(Strong Convergence Criterion for Power Series) Let* (a_n) *be a sequence in* $\mathcal{R}$ *(resp.* $\mathcal{C}$*), and let*

$$\lambda_0 = \limsup_{n\to\infty}\left(\frac{-\lambda(a_n)}{n}\right) \text{ in } \mathbb{R}\cup\{-\infty,\infty\}.$$

Let $x_0 \in \mathcal{R}$ *(resp.* $\mathcal{C}$*) be fixed and let* $x \in \mathcal{R}$ *(resp.* $\mathcal{C}$*) be given. Then the power series* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *converges strongly if* $\lambda(x-x_0) > \lambda_0$ *and is strongly divergent if* $\lambda(x-x_0) < \lambda_0$ *or if* $\lambda(x-x_0) = \lambda_0$ *and* $-\lambda(a_n)/n > \lambda_0$ *for infinitely many* n.

The following two examples show that for the case when $\lambda(x-x_0) = \lambda_0$ and $-\lambda(a_n)/n \geq \lambda_0$ for only finitely many n, the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ can either converge or diverge strongly. In this case, Theorem 3.4 provides a test for weak convergence.

Example: For each $n \geq 0$, let $a_n = d$; and let $x_0 = 0$ and $x = 1$. Then $\lambda_0 = \limsup_{n\to\infty}(-1/n) = 0 = \lambda(x)$. Moreover, we have that $-\lambda(a_n)/n = -1/n < \lambda_0$ for all $n \geq 0$; and $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} d$ is strongly divergent.

Example: For each n, let $q_n \in \mathbb{Q}$ be such that $\sqrt{n}/2 < q_n < \sqrt{n}$, let $a_n = d^{q_n}$; and let $x_0 = 0$ and $x = 1$. Then $\lambda_0 = \limsup_{n\to\infty}(-q_n/n) = 0 = \lambda(x)$. Moreover, we have that $-\lambda(a_n)/n = -q_n/n < 0 = \lambda_0$ for all $n \geq 0$; and $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} d^{q_n}$ converges strongly since the sequence (d^{q_n}) is a null sequence with respect to the strong topology.

Remark: Let x_0 and λ_0 be as in Theorem 3.3, and let $x \in \mathcal{R}$ (resp. $\mathcal{C}$) be such that $\lambda(x-x_0) = \lambda_0$. Then $\lambda_0 \in \mathbb{Q}\cup\{\infty\}$. But if $\lambda_0 = \infty$, then $x = x_0$ and hence $\sum_{n=0}^{\infty} a_n(x-x_0)^n = 0$. So it remains to discuss the case when $\lambda(x-x_0) = \lambda_0 \in \mathbb{Q}$.

Theorem 3.4: *(Weak Convergence Criterion for Power Series) Let* (a_n) *be a sequence in* $\mathcal{R}$ *(resp.* $\mathcal{C}$*), and let* $\lambda_0 = \limsup_{n\to\infty}(-\lambda(a_n)/n) \in \mathbb{Q}$. *Let* $x_0 \in \mathcal{R}$ *(resp.* $\mathcal{C}$*) be fixed, and let* $x \in \mathcal{R}$ *(resp.* $\mathcal{C}$*) be such that* $\lambda(x-x_0) = \lambda_0$. *For each* $n \geq 0$, *let* $b_n = a_n d^{n\lambda_0}$. *Suppose that the sequence* (b_n) *is regular and write* $\bigcup_{n=0}^{\infty} supp(b_n) = \{q_1, q_2, \ldots\}$; *with* $q_{j_1} < q_{j_2}$ *if* $j_1 < j_2$. *For each* n, *write* $b_n = \sum_{j=1}^{\infty} b_{n_j} d^{q_j}$, *where* $b_{n_j} = b_n[q_j]$. *Let* $r = 1/\sup\left\{\limsup_{n\to\infty}|b_{n_j}|^{1/n} : j \geq 1\right\}$ *in* $\mathbb{R}\cup\{\infty\}$, *with the conventions*

$1/0 = \infty$ and $1/\infty = 0$. Then $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely weakly if $|(x-x_0)[\lambda_0]| < r$ and is weakly divergent if $|(x-x_0)[\lambda_0]| > r$.

Corollary 3.5: *(Power Series with Purely Real or Complex Coefficients) Let $\sum_{n=0}^{\infty} a_n X^n$ be a power series with purely real (resp. complex) coefficients and with classical radius of convergence equal to η. Let $x \in \mathcal{R}$ (resp. $\mathcal{C}$), and let $A_n(x) = \sum_{j=0}^{n} a_j x^j \in \mathcal{R}$ (resp. $\mathcal{C}$). Then, for $|x| < \eta$ and $|x| \not\approx \eta$, the sequence $(A_n(x))$ converges absolutely weakly. We define the limit to be the continuation of the power series to $\mathcal{R}$ (resp. $\mathcal{C}$).*

Thus, we can now extend real and complex functions representable by power series to the Levi-Civita fields $\mathcal{R}$ and $\mathcal{C}$.

Définition: The series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}, \ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \ \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \text{ and } \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

converge absolutely weakly in $\mathcal{R}$ and $\mathcal{C}$ for any x, at most finite in absolute value. We define these series to be $\exp(x)$, $\cos(x)$, $\sin(x)$, $\cosh(x)$ and $\sinh(x)$ respectively.

Remark: For x in $\mathcal{R}$ (resp. $\mathcal{C}$), infinitely small in absolute value, the series above converge strongly in $\mathcal{R}$ (resp. $\mathcal{C}$), as shown in [14]. The assertion also follows readily from Theorem 3.3.

A detailed study of the transcendental functions can be found in [15]. In particular, it is easily shown that addition theorems similar to the real ones hold for these functions.

3.2 Differentiability and Re-expandability

We begin this section by defining differentiability.

Définition: Let $D \subset \mathcal{R}$ (resp. $\mathcal{C}$) be open and let $f : D \to \mathcal{R}$ (resp. $\mathcal{C}$). Then we say that f is differentiable at $x_0 \in D$ if there exists a number $f'(x_0) \in \mathcal{R}$ (resp. $\mathcal{C}$), called the derivative of f at x_0, such that for every $\epsilon > 0$ in $\mathcal{R}$, there exists $\delta > 0$ in $\mathcal{R}$ such that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \epsilon \text{ for all } x \in D \text{ satisfying } 0 < |x - x_0| < \delta.$$

Moreover, we say that f is differentiable on D if f is differentiable at every point in D.

Theorem 3.6: *Let* $x_0 \in \mathcal{R}$ *(resp.* $\mathcal{C}$*) be given, let* (a_n) *be a sequence in* $\mathcal{R}$ *(resp.* $\mathcal{C}$*), let* $\lambda_0 = \limsup_{n\to\infty} \left(-\lambda\left(a_n\right)/n\right) \in \mathbb{Q}$*; and for all* $n \geq 0$ *let* $b_n = d^{n\lambda_0} a_n$*. Suppose that the sequence* (b_n) *is regular; and write* $\bigcup_{n=0}^{\infty} supp(b_n) = \{q_1, q_2, \ldots\}$ *with* $q_{j_1} < q_{j_2}$ *if* $j_1 < j_2$*. For all* $n \geq 0$*, write* $b_n = \sum_{j=1}^{\infty} b_{n_j} d^{q_j}$ *where* $b_{n_j} = b_n\left[q_j\right]$*; and let*

$$\eta = \frac{1}{\sup\left\{\limsup_{n\to\infty} |b_{n_j}|^{1/n} : j \geq 1\right\}} \text{ in } \mathbb{R} \cup \{\infty\}. \tag{3.3}$$

Then, for all $\sigma \in \mathbb{R}$ *satisfying* $0 < \sigma < \eta$*, the function* $f : B\left(x_0, \sigma d^{\lambda_0}\right) \to \mathcal{R}$ *(resp.* $\mathcal{C}$*), given by* $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$*, under weak convergence, is infinitely often differentiable on the ball* $B\left(x_0, \sigma d^{\lambda_0}\right)$*, and the derivatives are given by* $f^{(k)}(x) = g_k(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, a_n (x - x_0)^{n-k}$ *for all* $x \in B\left(x_0, \sigma d^{\lambda_0}\right)$ *and for all* $k \geq 1$*. In particular, we have that* $a_k = f^{(k)}(x_0)/k!$ *for all* $k = 0, 1, 2, \ldots$*.*

Proof: As in the proof of Theorem 3.4 (see [19] for that proof), we may assume that $\lambda_0 = 0$, $b_n = a_n$ for all $n \geq 0$, and $\min\left(\bigcup_{n=0}^{\infty} \operatorname{supp}(a_n)\right) = 0$.

Using induction on k, it suffices to show that the result is true for $k = 1$. Since $\lim_{n\to\infty} n^{1/n} = 1$ and since $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges weakly for $x \in B(x_0, \sigma)$, we obtain that $\sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$ converges weakly for $x \in B(x_0, \sigma)$. Next we show that f is differentiable at x with derivative $f'(x) = g_1(x)$ for all $x \in B(x_0, \sigma)$; it suffices to show that there exists $M \in \mathcal{R}$ such that

$$\left|\frac{f(x+h) - f(x)}{h} - g_1(x)\right| < M|h| \tag{3.4}$$

for all $x \in B(x_0, \sigma)$ and for all $h \neq 0$ in $\mathcal{R}$ (resp. $\mathcal{C}$) satisfying $x + h \in B(x_0, \sigma)$.

We show that Equation (3.4) holds for $M = d^{-1}$. First let $|h|$ be finite. Since $f(x)$, $f(x+h)$ and $g_1(x)$ are all at most finite in absolute value, we obtain that

$$\lambda\left(\frac{f(x+h) - f(x)}{h} - g_1(x)\right) \geq 0.$$

On the other hand, we have that $\lambda\left(d^{-1}|h|\right) = -1 + \lambda(h) = -1$. Hence Equation (3.4) holds.

Now let $|h|$ be infinitely small. Write $h = h_0 d^r (1 + h_1)$ with $h_0 \in \mathbb{R}$ (resp. $\mathbb{C}$), $0 < r \in \mathbb{Q}$ and $0 \leq |h_1| \ll 1$. Let $s \leq 2r$ be given. Since (a_n) is regular, there exist only finitely many elements in $[0, s] \cap \bigcup_{n=0}^{\infty} \mathrm{supp}(a_n)$; write $[0, s] \cap \bigcup_{n=0}^{\infty} \mathrm{supp}\,(a_n) = \{q_{1,s}, q_{2,s}, \ldots, q_{j,s}\}$. Thus,

$$
\begin{aligned}
f(x+h)[s] &= \sum_{n=0}^{\infty} \left(\sum_{l=1}^{j} a_n [q_{l,s}] (x + h - x_0)^n [s - q_{l,s}] \right) \\
&= \sum_{l=1}^{j} \left(\sum_{n=0}^{\infty} a_n [q_{l,s}] \sum_{\nu=0}^{n} \left(\frac{n!}{\nu! (n-\nu)!} h^{\nu} (x - x_0)^{n-\nu} \right) [s - q_{l,s}] \right) \\
&= \sum_{l=1}^{j} \left(\begin{array}{c} \sum_{n=0}^{\infty} a_n [q_{l,s}] (x - x_0)^n [s - q_{l,s}] \\ + \sum_{n=1}^{\infty} n a_n [q_{l,s}] \left(h (x - x_0)^{n-1} \right) [s - q_{l,s}] \\ + \sum_{n=2}^{\infty} \frac{n(n-1)}{2} a_n [q_{l,s}] \left(h^2 (x - x_0)^{n-2} \right) [s - q_{l,s}] \end{array} \right).
\end{aligned}
$$

Other terms are not relevant (they are all equal to 0), since the corresponding powers of h are infinitely smaller than d^s in absolute value, and hence infinitely smaller than $d^{s-q_{l,s}}$ for all $l \in \{1, \ldots, j\}$. Thus

$$
\begin{aligned}
f(x+h)[s] &= \sum_{n=0}^{\infty} \left(\sum_{l=1}^{j} a_n [q_{l,s}] (x - x_0)^n [s - q_{l,s}] \right) \\
&\quad + \sum_{n=1}^{\infty} \left(\sum_{l=1}^{j} n a_n [q_{l,s}] \left(h (x - x_0)^{n-1} \right) [s - q_{l,s}] \right) \\
&\quad + \sum_{n=2}^{\infty} \left(\sum_{l=1}^{j} \frac{n(n-1)}{2} a_n [q_{l,s}] \left(h^2 (x - x_0)^{n-2} \right) [s - q_{l,s}] \right) \\
&= \sum_{n=0}^{\infty} \left(a_n (x - x_0)^n \right) [s] + \sum_{n=1}^{\infty} \left(n h a_n (x - x_0)^{n-1} \right) [s] \\
&\quad + \sum_{n=2}^{\infty} \left(\frac{n(n-1)}{2} h^2 a_n (x - x_0)^{n-2} \right) [s].
\end{aligned}
$$

Therefore, we obtain that

$$
\frac{f(x+h) - f(x)}{h} - g_1(x) =_r h_0 d^r \sum_{n=2}^{\infty} \frac{n(n-1)}{2} a_n (x - x_0)^{n-2}. \tag{3.5}
$$

Since $\lambda(a_n) \geq 0$ for all $n \geq 2$ and since $\lambda(x - x_0) \geq 0$, we obtain that

$$\lambda\left(\sum_{n=2}^{\infty} \frac{n(n-1)}{2} a_n (x - x_0)^{n-2}\right) \geq 0.$$

Thus, Equation (3.5) entails that

$$\lambda\left(\frac{f(x+h) - f(x)}{h} - g_1(x)\right) \geq r = \lambda(h) > \lambda(h) - 1 = \lambda\left(d^{-1}|h|\right);$$

and hence Equation (3.4) holds. □

Remark: It is shown in [15] that the condition in Equation 3.4 entails the differentiability of the function f at x with derivative $f'(x) = g(x)$. This covers all the cases of topological differentiability (ϵ-δ definition above), equidifferentiability [2, 5] as well as the differentiability based on the derivates [4, 15].

The following result shows that, like in $\mathbb{R}$ and $\mathbb{C}$, power series on $\mathcal{R}$ and $\mathcal{C}$ can be re-expanded around any point within their domain of convergence.

Theorem 3.7: *Let $x_0 \in \mathcal{R}$ (resp. $\mathcal{C}$) be given, let (a_n) be a regular sequence in $\mathcal{R}$ (resp. $\mathcal{C}$), with $\lambda_0 = \limsup_{n\to\infty} \{-\lambda(a_n)/n\} = 0$; and let $\eta \in \mathbb{R}$ be the radius of weak convergence of $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, given by Equation (3.3). Let $y_0 \in \mathcal{R}$ (resp. $\mathcal{C}$) be such that $|(y_0 - x_0)[0]| < \eta$. Then, for all $x \in \mathcal{R}$ (resp. $\mathcal{C}$) satisfying $|(x - y_0)[0]| < \eta - |(y_0 - x_0)[0]|$, we have that $\sum_{k=0}^{\infty} f^{(k)}(y_0)/(k!)(x - y_0)^k$ converges weakly to $f(x)$; and the radius of convergence is exactly $\eta - |(y_0 - x_0)[0]|$.*

Proof: Let x be such that $|(x - y_0)[0]| < \eta - |(y_0 - x_0)[0]|$. Since $|(y_0 - x_0)[0]| < \eta$, we have that

$$f^{(k)}(y_0) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (y_0 - x_0)^{n-k} \text{ for all } k \geq 0.$$

Since $|(x - y_0)[0]| < \eta - |(y_0 - x_0)[0]|$, we obtain that

$$|(x - x_0)[0]| \leq |(x - y_0)[0]| + |(y_0 - x_0)[0]| < \eta.$$

Hence $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ converges absolutely weakly in $\mathcal{R}$ (resp. $\mathcal{C}$).

Now let $q \in \mathbb{Q}$ be given. Then

$$\begin{aligned} f(x)[q] &= \left(\sum_{n=0}^{\infty} a_n (x-x_0)^n\right)[q] = \left(\sum_{n=0}^{\infty} a_n (y_0 - x_0 + x - y_0)^n\right)[q] \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{n\ldots(n-k+1)}{k!} a_n (y_0-x_0)^{n-k}(x-y_0)^k\right)[q]\,. \quad (3.6) \end{aligned}$$

Because of absolute convergence in $\mathbb{R}$ (resp. $\mathbb{C}$), we can interchange the order of the sums in Equation (3.6) to obtain

$$\begin{aligned} f(x)[q] &= \left(\sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{n=k}^{\infty} n\ldots(n-k+1)\, a_n (y_0-x_0)^{n-k}\right)(x-y_0)^k\right)[q] \\ &= \left(\sum_{k=0}^{\infty} \frac{f^{(k)}(y_0)}{k!}(x-y_0)^k\right)[q]\,. \end{aligned}$$

Thus, for all $q \in \mathbb{Q}$, we have that $\left(\sum_{k=0}^{\infty} f^{(k)}(y_0)/k!\,(x-y_0)^k\right)[q]$ converges in $\mathbb{R}$ (resp. $\mathbb{C}$) to $f(x)[q]$.

Consider the sequence $(A_m)_{m\geq 1}$, where $A_m = \sum_{k=0}^{m} f^{(k)}(y_0)/k!\,(x-y_0)^k$ for each $m \geq 1$. Since (a_n) is regular and since $\lambda(y_0-x_0) \geq 0$, we obtain that the sequence $\left(f^{(k)}(y_0)\right)$ is regular. Since, in addition, $\lambda(x-y_0) \geq 0$, we obtain that the sequence (A_m) itself is regular. Since (A_m) is regular and since $(A_m[q])$ converges in $\mathbb{R}$ (resp. $\mathbb{C}$) to $f(x)[q]$ for all $q \in \mathbb{Q}$, we finally obtain that (A_m) converges weakly to $f(x)$; and we can write $\sum_{k=0}^{\infty} f^{(k)}(y_0)/k!\,(x-y_0)^k = f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ for all x satisfying $|(x-y_0)[0]| < \eta - |(y_0-x_0)[0]|$.

Next we show that $\eta - |(y_0-x_0)[0]|$ is indeed the radius of weak convergence of $\sum_{k=0}^{\infty} f^{(k)}(y_0)/(k!)(x-y_0)^k$. So let $r > \eta - |(y_0-x_0)[0]|$ be given in $\mathbb{R}$; we will show that there exists $x \in \mathcal{R}$ (resp. $\mathcal{C}$) satisfying $|(x-y_0)[0]| < r$ such that the power series $\sum_{k=0}^{\infty} f^{(k)}(y_0)/(k!)(x-y_0)^k$ is weakly divergent. If $(y_0-x_0)[0] = 0$, let $x = y_0 + (r+\eta)/2$. Then we obtain that

$$|(x-y_0)[0]| = \frac{r+\eta}{2}[0] = \frac{r+\eta}{2} < r.$$

But

$$|(x-x_0)[0]| = |(x-y_0)[0] + (y_0-x_0)[0]| = |(x-y_0)[0]| = \frac{r+\eta}{2} > \eta;$$

and hence $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is weakly divergent.

On the other hand, if $(y_0 - x_0)[0] \neq 0$, let

$$x = y_0 + \frac{r + \eta - |(y_0 - x_0)[0]|}{2} \frac{(y_0 - x_0)[0]}{|(y_0 - x_0)[0]|}.$$

Then we obtain that

$$|(x - y_0)[0]| = \frac{r + \eta - |(y_0 - x_0)[0]|}{2} < r.$$

But

$$\begin{aligned} |(x - x_0)[0]| &= \left| (y_0 - x_0)[0] + \frac{r + \eta - |(y_0 - x_0)[0]|}{2} \frac{(y_0 - x_0)[0]}{|(y_0 - x_0)[0]|} \right| \\ &= \left| \frac{r + \eta + |(y_0 - x_0)[0]|}{2} \frac{(y_0 - x_0)[0]}{|(y_0 - x_0)[0]|} \right| \\ &= \frac{r + \eta + |(y_0 - x_0)[0]|}{2} > \eta; \end{aligned}$$

and hence $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is weakly divergent.

Thus, in both cases, we have that $|(x - y_0)[0]| < r$ and $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is weakly divergent. Hence there exists $t_0 \in \mathbb{Q}$ such that $\sum_{n=0}^{\infty} (a_n (x - x_0)^n)[t_0]$ diverges in $\mathbb{R}$ (resp. $\mathbb{C}$). It follows that $\sum_{k=0}^{\infty} \left(f^{(k)}(y_0)/k! \, (x - y_0)^k \right)[t_0]$ diverges in $\mathbb{R}$ (resp. $\mathbb{C}$) and therefore that $\sum_{k=0}^{\infty} f^{(k)}(y_0)/k! \, (x - y_0)^k$ is weakly divergent. So $\eta - |(y_0 - x_0)[0]|$ is the radius of weak convergence of $\sum_{k=0}^{\infty} f^{(k)}(y_0)/(k!)(x - y_0)^k$. □

4 $\mathcal{R}$-Analytic Functions

In this section, we introduce a class of functions on $\mathcal{R}$ that are given locally by power series and for which all the common theorems of real calculus will be shown to hold.

4.1 Definition and Algebraic Properties

Définition: Let $a, b \in \mathcal{R}$ be such that $0 < b - a \sim 1$ and let $f : [a, b] \to \mathcal{R}$. Then we say that f is expandable or $\mathcal{R}$-analytic on $[a, b]$ if for all $x \in [a, b]$

there exists a finite $\delta > 0$ in $\mathcal{R}$, and there exists a regular sequence $(a_n(x))$ in $\mathcal{R}$ such that, under weak convergence, $f(y) = \sum_{n=0}^{\infty} a_n(x)(y-x)^n$ for all $y \in (x-\delta, x+\delta) \cap [a,b]$.

Définition: Let $a < b$ in $\mathcal{R}$ be such that $t = \lambda(b-a) \neq 0$ and let $f : [a,b] \to \mathcal{R}$. Then we say that f is $\mathcal{R}$-analytic on $[a,b]$ if the function $F : [d^{-t}a, d^{-t}b] \to \mathcal{R}$, given by $F(x) = f(d^t x)$, is $\mathcal{R}$-analytic on $[d^{-t}a, d^{-t}b]$.

Lemma 4.1: *Let $a, b \in \mathcal{R}$ be such that $0 < b - a \sim 1$, let $f, g : [a,b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a,b]$ and let $\alpha \in \mathcal{R}$ be given. Then $f + \alpha g$ and $f \cdot g$ are $\mathcal{R}$-analytic on $[a,b]$.*

Proof: The proof of the first part is straightforward; so we present here only the proof of the second part. Let $x \in [a,b]$ be given. Then there exist finite $\delta_1 > 0$ and $\delta_2 > 0$, and there exist regular sequences (a_n) and (b_n) in $\mathcal{R}$ such that $f(x+h) = \sum_{n=0}^{\infty} a_n h^n$ for $0 \leq |h| < \delta_1$ and $g(x+h) = \sum_{n=0}^{\infty} b_n h^n$ for $0 \leq |h| < \delta_2$. Let $\delta = \min\{\delta_1/2, \delta_2/2\}$. Then $0 < \delta \sim 1$. For each n, let $c_n = \sum_{j=0}^{n} a_j b_{n-j}$. Then the sequence (c_n) is regular. Since $\sum_{n=0}^{\infty} a_n h^n$ converges weakly for all h such that $x + h \in [a,b]$ and $0 \leq |h| < \delta_1$, so does $\sum_{n=0}^{\infty} a_n[t] h^n$ for all $t \in \bigcup_{n=0}^{\infty} \text{supp}(a_n)$. Hence $\sum_{n=0}^{\infty} |(a_n[t] h^n)[q]|$ converges in $\mathbb{R}$ for all $q \in \mathbb{Q}$, for all $t \in \bigcup_{n=0}^{\infty} \text{supp}(a_n)$ and for all h satisfying $x + h \in [a,b]$, $0 \leq |h| < 3\delta/2$ and $|h| \not\approx 3\delta/2$.

Now let $h \in \mathcal{R}$ be such that $x + h \in [a,b]$ and $0 \leq |h| < \delta$. Then

$$\begin{aligned} \sum_{n=0}^{\infty} |(a_n h^n)[q]| &= \sum_{n=0}^{\infty} \left| \sum_{\substack{q_1 \in \text{supp}(a_n),\, q_2 \in \text{supp}(h^n) \\ q_1 + q_2 = q}} a_n[q_1]\, h^n[q_2] \right| \\ &\leq \sum_{\substack{q_1 \in \bigcup_{n=0}^{\infty} \text{supp}(a_n),\, q_2 \in \bigcup_{n=0}^{\infty} \text{supp}(h^n) \\ q_1 + q_2 = q}} \sum_{n=0}^{\infty} |a_n[q_1]|\, |h^n[q_2]|. \end{aligned}$$

Since $\sum_{n=0}^{\infty} |a_n[q_1]|\, |h^n[q_2]|$ converges in $\mathbb{R}$ and since only finitely many terms contribute to the first sum by regularity, we obtain that $\sum_{n=0}^{\infty} |(a_n h^n)[q]|$ converges for each $q \in \mathbb{Q}$. Since $\sum_{n=0}^{\infty} a_n h^n$ converges absolutely weakly, since $\sum_{n=0}^{\infty} b_n h^n$ converges weakly and since the sequences $(\sum_{m=0}^{n} a_m h^m)$ and $(\sum_{m=0}^{n} b_m h^m)$ are both regular, we obtain that $\sum_{n=0}^{\infty} a_n h^n \cdot \sum_{n=0}^{\infty} b_n h^n = \sum_{n=0}^{\infty} c_n h^n$; hence $(f \cdot g)(x+h) = \sum_{n=0}^{\infty} c_n h^n$. This is true for all $x \in [a,b]$; hence $(f \cdot g)$ is $\mathcal{R}$-analytic on $[a,b]$. □

Corollary 4.2: *Let $a < b$ in $\mathcal{R}$ be given, let $f, g : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$ and let $\alpha \in \mathcal{R}$ be given. Then $f + \alpha g$ and $f \cdot g$ are $\mathcal{R}$-analytic on $[a, b]$.*

Proof: Let $t = \lambda(b - a)$, and let $F, G : [d^{-t}a, d^{-t}b]$ be given by $F(x) = f(d^t x)$ and $G(x) = g(d^t x)$. Then, by definition, F and G are both $\mathcal{R}$-analytic on $[d^{-t}a, d^{-t}b]$; and hence so are $F + \alpha G$ and $F \cdot G$. For all $x \in [d^{-t}a, d^{-t}b]$, we have that $(F + \alpha G)(x) = (f + \alpha g)(d^t x)$ and $(F \cdot G)(x) = (f \cdot g)(d^t x)$. Since $F + \alpha G$ and $F \cdot G$ are $\mathcal{R}$-analytic on $[d^{-t}a, d^{-t}b]$, so are $f + \alpha g$ and $f \cdot g$ on $[a, b]$. □

Lemma 4.3: *Let $a < b$ and $c < e$ in $\mathcal{R}$ be such that $b - a$ and $e - c$ are both finite. Let $f : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$, let $g : [c, e] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[c, e]$, and let $f([a, b]) \subset [c, e]$. Then $g \circ f$ is $\mathcal{R}$-analytic on $[a, b]$.*

Proof: Let $x \in [a, b]$ be given. There exist finite $\delta_1 > 0$ and $\delta_2 > 0$, and there exist regular sequences (a_n) and (b_n) in $\mathcal{R}$ such that

$$|h| < \delta_1 \text{ and } x + h \in [a, b] \quad \Rightarrow \quad f(x + h) = f(x) + \sum_{n=1}^{\infty} a_n h^n \text{ ; and}$$

$$|y| < \delta_2 \text{ and } f(x) + y \in [c, e] \quad \Rightarrow \quad g(f(x) + y) = g(f(x)) + \sum_{n=1}^{\infty} b_n y^n.$$

Since $F(h) = \left(\sum_{n=1}^{\infty} a_n h^n\right)[0]$ is continuous on $\mathbb{R}$, we can choose $\delta \in (0, \delta_1/2]$ such that $|h| < \delta$ and $x + h \in [a, b] \Rightarrow \left|\sum_{n=1}^{\infty} a_n h^n\right| < \delta_2/2$. Thus, for $|h| < \delta$ and $x + h \in [a, b]$, we have that

$$\begin{aligned}(g \circ f)(x + h) &= g\left(f(x) + \sum_{n=1}^{\infty} a_n h^n\right) = g(f(x)) + \sum_{k=1}^{\infty} b_k \left(\sum_{n=1}^{\infty} a_n h^n\right)^k \\ &= (g \circ f)(x) + \sum_{k=1}^{\infty} b_k \left(\sum_{n=1}^{\infty} a_n h^n\right)^k . \end{aligned} \tag{4.7}$$

For each k, let $V_k(h) = b_k \left(\sum_{n=1}^{\infty} a_n h^n\right)^k$. Then $V_k(h)$ is an infinite series $V_k(h) = \sum_{j=1}^{\infty} a_{kj} h^j$, where the sequence (a_{kj}) is regular in $\mathcal{R}$ for each k. By our choice of δ, we have that for all $q \in \mathbb{Q}$, $\sum_{j=1}^{\infty} \left|(a_{kj} h^j)[q]\right|$ converges in $\mathbb{R}$; so we can rearrange the terms in $V_k(h)[q] = \sum_{j=1}^{\infty} (a_{kj} h^j)[q]$. Moreover, the double sum $\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left(a_{kj}h^j\right)[q]$ converges; so we can interchange the order of the summations (see for example [11] pp 205-208) and obtain that

$$\begin{aligned}((g\circ f)(x+h))[q] &= ((g\circ f)(x))[q]+\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\left(a_{kj}h^j\right)[q]\\ &= ((g\circ f)(x))[q]+\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(a_{kj}h^j\right)[q]\end{aligned}$$

for all $q\in\mathbb{Q}$. Therefore,

$$(g\circ f)(x+h)=(g\circ f)(x)+\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}a_{kj}h^j=(g\circ f)(x)+\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{kj}h^j.$$

Thus, rearranging and regrouping the terms in Equation (4.7), we obtain that $(g\circ f)(x+h)=(g\circ f)(x)+\sum_{j=1}^{\infty}c_jh^j$, where the sequence (c_j) is regular.
□

Just as we did in generalizing Lemma 4.1 to Corollary 4.2, we can now generalize Lemma 4.3 to infinitely small and infinitely large domains and obtain the following result.

Corollary 4.4: *Let $a<b$ and $c<e$ in $\mathcal{R}$ be given. Let $f:[a,b]\to\mathcal{R}$ be $\mathcal{R}$-analytic on $[a,b]$, let $g:[c,e]\to\mathcal{R}$ be $\mathcal{R}$-analytic on $[c,e]$, and let $f([a,b])\subset[c,e]$. Then $g\circ f$ is $\mathcal{R}$-analytic on $[a,b]$.*

Proof: Let $t=\lambda(b-a)$ and $j=\lambda(e-c)$; and let $F:[d^{-t}a,d^{-t}b]\to\mathcal{R}$ and $G:[d^{-j}c,d^{-j}e]\to\mathcal{R}$ be given by

$$F(x)=d^{-j}f\left(d^tx\right)\text{ and }G(x)=g\left(d^jx\right).$$

Then F and G are $\mathcal{R}$-analytic on $[d^{-t}a,d^{-t}b]$ and $[d^{-j}c,d^{-j}e]$, respectively; moreover, $F\left([d^{-t}a,d^{-t}b]\right)\subset[d^{-j}c,d^{-j}e]$. Since $[d^{-t}a,d^{-t}b]$ and $[d^{-j}c,d^{-j}e]$ both have finite lengths, by our choice of t and j, we obtain by the previous theorem that $G\circ F$ is $\mathcal{R}$-analytic on $[d^{-t}a,d^{-t}b]$. But for all $x\in[d^{-t}a,d^{-t}b]$, we have that

$$G\circ F(x)=G(F(x))=G\left(d^{-j}f\left(d^tx\right)\right)=g\left(f\left(d^tx\right)\right)=g\circ f\left(d^tx\right).$$

Since $G\circ F$ is $\mathcal{R}$-analytic on $[d^{-t}a,d^{-t}b]$, it follows that $g\circ f$ is $\mathcal{R}$-analytic on $[a,b]$.
□

Lemma 4.5: *Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$. Then f is bounded on $[a, b]$.*

Proof: Let $F : [0, 1] \to \mathcal{R}$ be given by

$$F(x) = f((b-a)x + a) - \frac{f(a) + f(b)}{2}.$$

Then F is $\mathcal{R}$-analytic on $[0, 1]$ by Corollary 4.4 and Corollary 4.2; moreover, f is bounded on $[a, b]$ if and only if F is bounded on $[0, 1]$. Thus, it suffices to show that F is bounded on $[0, 1]$.

For all $X \in [0, 1] \cap \mathbb{R}$ there exists a real $\delta(X) > 0$ and there exists a regular sequence $(a_n(X))$ in $\mathcal{R}$ such that $F(x) = \sum_{n=0}^{\infty} a_n(X)(x - X)^n$ for all $x \in (X - \delta(X), X + \delta(X)) \cap [0, 1]$. Thus, we obtain a real open cover, $\{(X - \delta(X)/2, X + \delta(X)/2) \cap \mathbb{R} : X \in [0, 1] \cap \mathbb{R}\}$, of the compact real set $[0, 1] \cap \mathbb{R}$. Therefore, there exists a positive integer m and there exist $X_1, \ldots, X_m \in [0, 1] \cap \mathbb{R}$ such that

$$[0, 1] \cap \mathbb{R} \subset \bigcup_{j=1}^{m} \left(\left(X_j - \frac{\delta(X_j)}{2}, X_j + \frac{\delta(X_j)}{2} \right) \cap \mathbb{R} \right).$$

It follows that $[0, 1] \subset \bigcup_{j=1}^{m} (X_j - \delta(X_j), X_j + \delta(X_j))$. Let

$$l = \min_{1 \le j \le m} \left\{ \min \left\{ \bigcup_{n=0}^{\infty} \operatorname{supp}(a_n(X_j)) \right\} \right\}.$$

Then $|F(x)| < d^{l-1}$ for all $x \in [0, 1]$, and hence F is bounded on $[0, 1]$. □

Remark: In the proof of Lemma 4.5, l is independent of the choice of the cover of $[0, 1] \cap \mathbb{R}$. It depends only on a, b, and f (or, in other words, on F); we will call it the index of f on $[a, b]$ and we will denote it by $i(f)$. Moreover, $\lambda(F(X)) = i(f)$ a.e. on $[0, 1] \cap \mathbb{R}$ and the same is true in the infinitely small neighborhood of any such X.

Proof: Let $X_1, \ldots, X_m$ and l be as in the proof of Lemma 4.5. Let $Z_1, \ldots, Z_k$ in $[0, 1] \cap \mathbb{R}$, let $\{(Z_j - \delta(Z_j), Z_j + \delta(Z_j)) \cap \mathbb{R} : 1 \le j \le k\}$ be an open cover of $[0, 1] \cap \mathbb{R}$, with $\delta(Z_j) > 0$ and real for all $j \in \{1, \ldots, k\}$, and let $l_1 = \min_{1 \le j \le k} \{\min \{\bigcup_{n=0}^{\infty} \operatorname{supp}(a_n(Z_j))\}\}$. Suppose $l_1 \ne l$. Without loss of generality, we may assume that $l < l_1$. In particular, $l < \infty$. Define F_R :

$[0,1]\cap\mathbb{R}\to\mathbb{R}$ by $F_R(Y)=F(Y)[l]$. For $Y\in(X_j-\delta(X_j),X_j+\delta(X_j))\cap[0,1]\cap\mathbb{R}$, we have that

$$F_R(Y)=\left(\sum_{n=0}^{\infty}a_n(X_j)(Y-X_j)^n\right)[l]=\sum_{n=0}^{\infty}a_n(X_j)[l](Y-X_j)^n. \quad (4.8)$$

Thus F_R is $\mathbb{R}$-analytic on $[0,1]\cap\mathbb{R}$. Moreover, $F_R(Y)=F(Y)[l]=0$ for all $Y\in\left(Z_1-\frac{\delta(Z_1)}{2},Z_1+\frac{\delta(Z_1)}{2}\right)\cap[0,1]\cap\mathbb{R}$. Using the identity theorem for analytic real functions, we obtain that $F_R(Y)=0$ for all $Y\in[0,1]\cap\mathbb{R}$. Using Equation (4.8), we obtain that $a_n(X_j)[l]=0$ for all $n\in\mathbb{N}\cup\{0\}$ and for all $j\in\{1,\ldots,m\}$, which contradicts the definition of l. Thus $l_1=l$.

Now let $x\in[0,1]$ be given. Then there exists $j\in\{1,\ldots,m\}$ such that $x\in(X_j-\delta(X_j),X_j+\delta(X_j))$, and hence $F(x)=\sum_{n=0}^{\infty}a_n(X_j)(x-X_j)^n$, where $\lambda(a_n(X_j))\geq l$ for all $n\geq 0$ and where $\lambda(x-X_j)\geq 0$. Thus $\lambda(F(x))\geq l$ for all $x\in[0,1]$. Moreover, $F_R(X)=F(X)[l]\neq 0$ for all but countably many $X\in[0,1]\cap\mathbb{R}$. Thus $\lambda(F(X))=l=i(f)$ a.e. on $[0,1]\cap\mathbb{R}$. Furthermore, if $X\in[0,1]\cap\mathbb{R}$ satisfies $\lambda(F(X))=l$ and if $x\in[0,1]$ satisfies $|x-X|\ll 1$, then $F(x)=F(X)+\sum_{n=1}^{\infty}a_n(X)(x-X)^n$, where $\lambda(a_n(X))\geq l$ for all $n\geq 1$ and where $\lambda(x-X)>0$. It follows that $f(x)\approx F(X)$; in particular, $\lambda(F(x))=\lambda(F(X))=l=i(f)$. This proves the last statement in the remark. □

Based on the discussion in the last paragraph, we immediately obtain the following result.

Corollary 4.6: *Let a,b,f and F be as in Lemma 4.5 and let $i(f)$ be the index of f on $[a,b]$. Then $i(f)=\min\{supp(F(x)):x\in[0,1]\}$.*

4.2 Calculus on the $\mathcal{R}$-Analytic Functions

In this section, we show that the $\mathcal{R}$-analytic functions satisfy the intermediate value theorem and they are infinitely often differentiable in their domain.

Theorem 4.7: *(Intermediate Value Theorem) Let $a<b$ in $\mathcal{R}$ be given and let $f:[a,b]\to\mathcal{R}$ be $\mathcal{R}$-analytic on $[a,b]$. Then f assumes on $[a,b]$ every intermediate value between $f(a)$ and $f(b)$.*

Proof: If $f(a)=f(b)$, there is nothing to prove, so we may assume that $f(a)\neq f(b)$. Let $F:[0,1]\to\mathcal{R}$ be as in the proof of Lemma 4.5. Then F is

$\mathcal{R}$-analytic on $[0,1]$; and f assumes on $[a,b]$ every intermediate value between $f(a)$ and $f(b)$ if and only if F assumes on $[0,1]$ every intermediate value between $F(0) = (f(a) - f(b))/2$ and $F(1) = (f(b) - f(a))/2 = -F(0)$. So without loss of generality, we may assume that $a = 0$, $b = 1$, and $f = F$. Also, since scaling the function by a constant factor does not affect the existence of intermediate values, we may assume that f has a zero index on $[a,b] = [0,1]$; that is, $i(f) = 0$.

Now let S be between $f(0)$ and $f(1)$. Without loss of generality, we may assume that $f(0) < 0 = S < f(1)$. Let $f_R : [0,1] \cap \mathbb{R} \to \mathbb{R}$ be given by $f_R(X) = f(X)[0]$. Since f_R is continuous on $[0,1] \cap \mathbb{R}$ (it being $\mathbb{R}$-analytic there), there exists $X \in [0,1] \cap \mathbb{R}$ such that $f_R(X) = 0$. Let $B = \{X \in [0,1] \cap \mathbb{R} : f_R(X) = 0\}$. Then $B \neq \emptyset$. If there exists $X \in B$ such that $f(X) = 0$, then we are done. So we may assume that $f(X) \neq 0$ for all $X \in B$.

First Claim: There exists $X_0 \in B$ such that for all finite $\Delta > 0$ $\exists$ $x \in (X_0 - \Delta, X_0 + \Delta) \cap [0,1]$ with $\lambda(x - X_0) = 0$ such that $f(x)/f(X_0) < 0$.
Proof of the first claim: Suppose not. Then for all $X \in B$ there exists $\Delta(X) > 0$, finite in $\mathcal{R}$, such that

$$\frac{f(x)}{f(X)} \geq 0 \, \forall \, x \in (X - \Delta(X), X + \Delta(X)) \cap [0,1] \text{ with } \lambda(x - X) = 0. \quad (4.9)$$

Since f_R is continuous on $[0,1] \cap \mathbb{R}$, we have that for all $Y \in ([0,1] \cap \mathbb{R}) \setminus B$ there exists a real $\Delta(Y) > 0$ such that $f_R(X)/f_R(Y) > 0$ for all $X \in (Y - 2\Delta(Y), Y + 2\Delta(Y)) \cap [0,1] \cap \mathbb{R}$. It follows that, for all $Y \in ([0,1] \cap \mathbb{R}) \setminus B$, $f(x)/f(Y) > 0$ for all $x \in (Y - \Delta(Y), Y + \Delta(Y)) \cap [0,1]$. In particular,

$$\frac{f(x)}{f(Y)} > 0 \, \forall \, x \in (Y - \Delta(Y), Y + \Delta(Y)) \cap [0,1] \text{ with } \lambda(x - Y) = 0. \quad (4.10)$$

Combining Equation (4.9) and Equation (4.10), we obtain that for all $X \in [0,1] \cap \mathbb{R}$ there exists a real $\delta(X) > 0$ such that

$$\frac{f(x)}{f(X)} \geq 0 \, \forall \, x \in (X - \delta(X), X + \delta(X)) \cap [0,1] \text{ with } \lambda(x - X) = 0. \quad (4.11)$$

$\{(X - \delta(X)/2, X + \delta(X)/2) \cap \mathbb{R} : X \in [0,1] \cap \mathbb{R}\}$ is a real open cover of the compact real set $[0,1] \cap \mathbb{R}$. Hence there exists a positive integer m and there exist $X_1, \ldots, X_m \in [0,1] \cap \mathbb{R}$ such that

$$[0,1] \cap \mathbb{R} \subset \bigcup_{j=1}^{m} \left(\left(X_j - \frac{\delta(X_j)}{2}, X_j + \frac{\delta(X_j)}{2} \right) \cap \mathbb{R} \right).$$

Thus $[0,1] \subset \bigcup_{j=1}^{m} (X_j - \delta(X_j), X_j + \delta(X_j))$.

By Equation (4.11), we have for $j \in \{1, \ldots, m\}$ that

$$\frac{f(x)}{f(X_j)} \geq 0 \ \forall \ x \in (X_j - \delta(X_j), X_j + \delta(X_j)) \cap [0,1] \text{ with } \lambda(x - X_j) = 0. \tag{4.12}$$

Using Equation (4.12), we obtain that $f(1)/f(0) \geq 0$, a contradiction to the fact that $f(0) < 0 < f(1)$. This finishes the proof of the first claim.

Since f is $\mathcal{R}$-analytic on $[0,1]$, there exists a real $\delta(X_0) > 0$ and there exists a regular sequence $(a_n(X_0))_{n \in \mathbb{N}}$ in $\mathcal{R}$ such that $f(X_0 + h) = f(X_0) + \sum_{n=1}^{\infty} a_n(X_0) h^n$ for $0 \leq |h| < \delta(X_0)$. Now we look for x such that $0 < |x| \ll 1$ and $f(X_0 + x) = S = 0$. That is we look for a root of the equation $f(X_0) + \sum_{n=1}^{\infty} a_n(X_0) x^n = 0$. Since $f_R(X_0) = 0$, we have that $0 < |f(X_0)| \ll 1$. Let $m = \min\{n \in \mathbb{N} : \lambda(a_n(X_0)) = 0\}$. Such an m exists by virtue of the remark that followed Lemma 4.5. Consider the polynomial

$$P(x) = f(X_0) + a_1(X_0) x + \cdots + a_{m-1}(X_0) x^{m-1} + a_m(X_0) x^m. \tag{4.13}$$

Thus,

$$f(X_0 + x) = P(x) + \sum_{n>m} a_n(X_0) x^n. \tag{4.14}$$

We distinguish two cases: $m > 1$ and $m = 1$.

Case I: $m > 1$. (m can be odd or even.)

Second Claim: $P(x)$ has a root $x_1 \in \mathcal{R}$ such that $X_0 + x_1 \in [0,1]$.

Proof of the second claim: Suppose not. Then $P(x)$ has the same sign as $P(0) = f(X_0)$, and hence

$$\frac{P(x)}{f(X_0)} > 0 \text{ for all } x \in \mathcal{R} \text{ satisfying } X_0 + x \in (X_0 - \delta(X_0), X_0 + \delta(X_0)) \cap [0,1]. \tag{4.15}$$

There exists $M_1 > 0$ and $M_2 > 0$ in $\mathbb{R}$ such that

$$|P(x)| > M_1 \text{ and } \left| \sum_{n>m} a_n(X_0) x^n \right| < M_2 |x|^{m+1}$$

for all $x \in \mathcal{R}$ satisfying $X_0 + x \in [X_0 - \delta(X_0)/2, X_0 + \delta(X_0)/2] \cap [0,1]$ and $\lambda(x) = 0$. Let

$$\delta_1 = \min\left\{ \left(\frac{M_1}{2M_2}\right)^{\frac{1}{m+1}}, \frac{\delta(X_0)}{2} \right\}.$$

Then $\delta_1 > 0$, δ_1 is finite, and

$$\left| \sum_{n>m} a_n (X_0) x^n \right| < M_2 |x|^{m+1} < \frac{M_1}{2} < \frac{|P(x)|}{2}$$

for all $x \in \mathcal{R}$ satisfying $X_0 + x \in [X_0 - \delta_1, X_0 + \delta_1] \cap [0,1]$ and $\lambda(x) = 0$. Thus $f(X_0 + x) = P(x) + \sum_{n>m} a_n(X_0) x^n$ has the same sign as $P(x)$ for all $x \in \mathcal{R}$ satisfying $X_0 + x \in [X_0 - \delta_1, X_0 + \delta_1] \cap [0,1]$ and $\lambda(x) = 0$. Since $\delta_1 < \delta(X_0)$, it follows from Equation (4.15) that $f(X_0 + x)/f(X_0) > 0$ for all $x \in \mathcal{R}$ satisfying $X_0 + x \in [X_0 - \delta_1, X_0 + \delta_1] \cap [0,1]$ and $\lambda(x) = 0$, which contradicts the result of the first claim. This finishes the proof of the second claim.

Since $\mathcal{C}$ is algebraically closed [2], $P(x)$ has exactly m roots in $\mathcal{C}$, including x_1 and not necessarily mutually distinct. We rewrite $P(x)$ as follows:

$$P(x) = a_m(X_0)(x - x_1)(x - x_2) \cdots (x - x_m), \tag{4.16}$$

where x_1 is as in the second claim above and where $x_2, \ldots, x_m$ are the other (not necessarily distinct) roots of $P(x)$ in $\mathcal{C}$. Since $\lambda(a_m(X_0)) = 0$, $\lambda(a_j(X_0)) > 0$ for all $j < m$, and $\lambda(f(X_0)) > 0$, it follows that $\lambda(x_j) > 0$ for $j = 1, 2, \ldots, m$. For if $\lambda(x) \leq 0$ then Equation (4.13) entails that $P(x) \approx a_m(X_0)x^m$ and hence $P(x) \neq 0$.

Définition: Let $Q(x)$ be a polynomial over $\mathcal{C}$ of degree n, let $\xi_1, \ldots, \xi_n$ be its n roots in $\mathcal{C}$, let $j \in \{1, \ldots, n\}$, and let $l \leq n$ be given in $\mathbb{N}$. Then we say that ξ_j has quasi-multiplicity l as a root of $Q(x)$ if, for some $j_1 < j_2 < \ldots < j_{l-1}$ in $\{1, \ldots, n\} \setminus \{j\}$, we have that

$$\xi_j \approx \xi_k \text{ if and only if } k \in \{j, j_1, j_2, \ldots, j_{l-1}\}.$$

Third Claim: At least one of the $\mathcal{R}$-roots of $P(x)$ has an odd quasi-multiplicity.
Proof of the third claim: Assume not. Then all $\mathcal{R}$-roots of $P(x)$ (including x_1) have even quasi-multiplicities. It follows that m is even and $a_m(X_0)$ has the same sign as $f(X_0)$. It follows that $P(x)$ and hence (as in the proof of the second claim) $f(X_0 + x)$ has the same sign as $f(X_0)$ for all x satisfying $\lambda(x) = 0$ and $X_0 + x \in (X_0 - \delta_1, X_0 + \delta_1) \cap [0,1]$ for some finite δ_1: $0 < \delta_1 \leq \delta(X_0)$. This contradicts the result of the first claim above.

Without loss of generality, we may assume that x_1 has an odd quasi-multiplicity, say l. Then $1 \leq l \leq m$.
Subcase I-1: $1 \leq l < m$. By rearranging the roots of $P(x)$, if necessary, we may assume that

$$x_1 \approx x_2 \ldots \approx x_l \text{ and } x_j \not\approx x_1 \text{ for } l < j \leq m. \tag{4.17}$$

Now we look for $y \in \mathcal{R}$ such that $\lambda(y) > \lambda(x_1)$ and

$$\begin{aligned} 0 &= f(X_0 + x_1 + y) \\ &= f(X_0 + x_1) + f'(X_0 + x_1)y + \ldots + \frac{f^{(l)}(X_0 + x_1)}{l!} y^l + \\ &\quad + \ldots + \frac{f^{(m)}(X_0 + x_1)}{m!} y^m + \sum_{k>m} \frac{f^{(k)}(X_0 + x_1)}{k!} y^k. \end{aligned}$$

It follows from Equations (4.16) and (4.17) that

$$P^{(l)}(x_1) \sim \prod_{j=l+1}^{m} (x_1 - x_j)\,;$$

and hence

$$\lambda\left(P^{(l)}(x_1)\right) = \sum_{j=l+1}^{m} \lambda(x_1 - x_j) \leq (m-l)\lambda(x_1). \tag{4.18}$$

Since $f^{(l)}(X_0 + x_1) = P^{(l)}(x_1) + \sum_{n>m} n \ldots (n-l+1) a_n (X_0)\, x_1^{n-l}$ and since

$$\begin{aligned} \lambda\left(\sum_{n>m} n \ldots (n-l+1) a_n (X_0)\, x_1^{n-l} \right) &\geq (m+1-l)\lambda(x_1) \\ &> (m-l)\lambda(x_1), \end{aligned}$$

it follows that

$$\lambda\left(f^{(l)}(X_0 + x_1)\right) = \lambda\left(P^{(l)}(x_1)\right) \leq (m-l)\lambda(x_1). \tag{4.19}$$

Let

$$g_1(y) := l! \frac{f(X_0 + x_1 + y)}{f^{(l)}(X_0 + x_1)}.$$

Then

$$g_1(y) = l! \frac{f(X_0 + x_1)}{f^{(l)}(X_0 + x_1)} + \sum_{k=1}^{l-1} \alpha_k y^k + y^l + \sum_{k>l} \alpha_k y^k \tag{4.20}$$

where

$$\alpha_k = \frac{l! f^{(k)}(X_0 + x_1)}{k! f^{(l)}(X_0 + x_1)}$$

for $k = 1, \ldots, l-1$ and for $k > l$.

Since $P(x_1) = 0$, it follows that

$$\lambda(f(X_0 + x_1)) = \lambda\left(\sum_{n>m} a_n(X_0)\, x_1^n\right) \geq (m+1)\lambda(x_1)$$

and hence

$$\lambda\left(l! \frac{f(X_0 + x_1)}{f^{(l)}(X_0 + x_1)}\right) \geq (m+1)\lambda(x_1) - (m-l)\lambda(x_1) = (l+1)\lambda(x_1). \quad (4.21)$$

For $1 \leq k < l$, we have, using Equations (4.16) and (4.17), that

$$\lambda\left(\frac{P^{(k)}(x_1)}{P^{(l)}(x_1)}\right) > (l-k)\lambda(x_1).$$

Also

$$\lambda\left(\frac{\sum_{n>m} n \ldots (n-k+1) a_n(X_0) x_1^{n-k}}{P^{(l)}(x_1)}\right) > (m-k)\lambda(x_1) - (m-l)\lambda(x_1) = (l-k)\lambda(x_1).$$

Hence

$$\begin{aligned} \lambda\left(\frac{f^{(k)}(X_0 + x_1)}{P^{(l)}(x_1)}\right) &= \lambda\left(\frac{P^{(k)}(x_1) + \sum_{n>m} n \ldots (n-k+1) a_n(X_0) x_1^{n-k}}{P^{(l)}(x_1)}\right) \\ &> (l-k)\lambda(x_1). \end{aligned}$$

Thus,

$$\begin{aligned} \lambda(\alpha_k) &= \lambda\left(\frac{f^{(k)}(X_0 + x_1)}{f^{(l)}(X_0 + x_1)}\right) = \lambda\left(\frac{f^{(k)}(X_0 + x_1)}{P^{(l)}(x_1)}\right) \\ &> (l-k)\lambda(x_1) \text{ for } 1 \leq k < l. \end{aligned} \quad (4.22)$$

Similarly, we show that

$$\lambda(\alpha_k) \geq (l-k)\lambda(x_1) \text{ for } l < k \leq m, \quad (4.23)$$

Finally, for $k > m$, we have that

$$\lambda\left(f^{(k)}(X_0+x_1)\right) = \lambda\left(\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(X_0)x_1^{n-k}\right) \geq 0.$$

Thus,

$$\begin{aligned}
\lambda(\alpha_k) &= \lambda\left(\frac{f^{(k)}(X_0+x_1)}{f^{(l)}(X_0+x_1)}\right) = \lambda\left(\frac{f^{(k)}(X_0+x_1)}{P^{(l)}(x_1)}\right) \\
&= \lambda\left(f^{(k)}(X_0+x_1)\right) - \lambda\left(P^{(l)}(x_1)\right) \geq 0 - (m-l)\lambda(x_1) \\
&\geq (l-m)\lambda(x_1) \text{ for } k > m. \qquad (4.24)
\end{aligned}$$

Let $z = y/x_1$ and $G_1(z) = g_1(x_1 z)/x_1^l$, where $g_1(y)$ is as in Equation (4.20). Then

$$G_1(z) = \beta_0 + \beta_1 z + \cdots + \beta_{l-1}z^{l-1} + z^l + \sum_{k>l} \beta_k z^k, \qquad (4.25)$$

where

$$\beta_0 = \frac{l!}{x_1^l}\frac{f(X_0+x_1)}{f^{(l)}(X_0+x_1)}$$

and $\beta_k = \alpha_k x_1^k/x_1^l = x_1^{k-l}\alpha_k$ for all $1 \leq k < l$ and $k > l$. Using Equations (4.21), (4.22), (4.23) and (4.24), we obtain that

$$\begin{aligned}
\lambda(\beta_0) &\geq (l+1)\lambda(x_1) - l\lambda(x_1) = \lambda(x_1) > 0; \\
\lambda(\beta_k) &= (k-l)\lambda(x_1) + \lambda(\alpha_k) > 0 \text{ for } 1 \leq k < l; \\
\lambda(\beta_k) &= (k-l)\lambda(x_1) + \lambda(\alpha_k) \geq 0 \text{ for } l < k \leq m; \\
\lambda(\beta_k) &\geq (k-l)\lambda(x_1) + (l-m)\lambda(x_1) \\
&= (k-m)\lambda(x_1) > 0 \text{ for } k > m.
\end{aligned}$$

Thus, $G_1(z)$ in Equation (4.25) is of the same form as $f(X_0+x)$ in Equation (4.14) except that the leading polynomial $Q_1(z) = \beta_0 + \beta_1 z + \cdots + z^l$ in Equation (4.25) has degree $l < m$, the degree of the leading polynomial $P(x)$ in Equation (4.14). Since l is odd, $Q_1(z)$ has at least one root $z_1 \in \mathcal{R}$ of odd quasi-multiplicity $l_1 \leq l$ and satisfying $\lambda(z_1) > 0$. It follows that

$$P_1(y) := l!\frac{f(X_0+x_1)}{f^{(l)}(X_0+x_1)} + \sum_{k=1}^{l-1}\alpha_k y^k + y^l = x_1^l Q_1\left(\frac{y}{x_1}\right)$$

has at least one root $y_1 \in \mathcal{R}$ of odd quasi-multiplicity $l_1 \leq l$ and satisfying $\lambda(y_1) = \lambda(x_1 z_1) > \lambda(x_1)$. Since $\lambda(y_1) > \lambda(x_1)$, we infer that $x_1 + y_1 \approx x_1$. Thus,

$$X_0 + x_1 + y_1 \text{ is on the same side from } X_0 \text{ as } X_0 + x_1. \tag{4.26}$$

Fourth Claim: $X_0 + x_1 + y_1 \in [0, 1]$.
Proof of the fourth claim: First assume that $X_0 \in (0, 1)$; then X_0 is finitely away from both 0 and 1. Since $|x_1 + y_1| \ll 1$, it follows that $X_0 + x_1 + y_1 \in (0, 1)$. Now assume that $X_0 = 0$. Since $X_0 + x_1 = x_1 \in [0, 1]$ by the second claim above, it follows that $0 < x_1 \ll 1$. Using Equation (4.26), it follows that $0 < x_1 + y_1 \ll 1$; and hence $X_0 + x_1 + y_1 = x_1 + y_1 \in (0, 1)$. Similarly, we show that if $X_0 = 1$ then $X_0 + x_1 + y_1 = 1 + x_1 + y_1 \in (0, 1)$. This finishes the proof of the fourth claim.

Continuing as above, we either obtain a root of f after finitely many iterations; or we have an infinite number of iterations, after a finite number of which, say N, the degree l_N of the leading polynomial will agree with the quasi-multiplicity of its roots for all the following iterations. Assume the latter situation happens. At the $(N + 2)$nd iteration (finding y_{N+1}), let

$$P_{N+1}(y) = \alpha_0^{(N+1)} + \sum_{k=1}^{l_N - 1} \alpha_k^{(N+1)} y^k + y^{l_N}$$

denote the leading polynomial, corresponding to $P_1(y)$ in Equation (4.20) of the second iteration and $y_{N+1} \in \mathcal{R}$ a root of $P_{N+1}(y)$ of quasi-multiplicity l_N. As in Equation (4.21), we have that

$$\lambda\left(\alpha_0^{(N+1)}\right) = \lambda(f(X_0 + x_1 + y_1 + \cdots + y_N)) \geq (l_N + 1)\lambda(y_N).$$

Since y_{N+1} has quasi-multiplicity l_N as a root of $P_{N+1}(y)$, it follows that

$$\begin{aligned} \alpha_0^{(N+1)} &= (-1)^{l_N} (\text{product of the roots of } P_{N+1}(y)) \\ &\approx (-1)^{l_N} y_{N+1}^{l_N} = -y_{N+1}^{l_N}. \end{aligned}$$

Hence

$$\lambda(y_{N+1}) = \frac{\lambda\left(\alpha_0^{(N+1)}\right)}{l_N} \geq \frac{l_N + 1}{l_N} \lambda(y_N) = \left(1 + \frac{1}{l_N}\right) \lambda(y_N) \geq \left(1 + \frac{1}{m}\right) \lambda(y_N).$$

Thus, we obtain a sequence (l_n) in $\mathbb{N}$ and a sequence (y_n) in $\mathcal{R}$ such that l_n is odd, $\lambda(y_{n+1}) > \lambda(y_n) > \lambda(x_1)$ and $X_0 + x_1 + \sum_{k=1}^{n} y_k \in [0,1]$ for all $n \geq 1$, and such that

$$\begin{aligned}
&l_{n+1} \leq l_n \leq m \text{ for all } n \geq 1 \text{ and } l_{n+1} = l_n \text{ for } n \geq N \\
&\lambda(y_{n+1}) \geq \left(1 + \frac{1}{l_n}\right)\lambda(y_n) \geq \left(1 + \frac{1}{m}\right)\lambda(y_n) \text{ for } n \geq N \\
&\lambda(f(X_0 + x_1 + y_1 + \cdots + y_n)) \geq (l_n + 1)\lambda(y_n) > \lambda(y_n).
\end{aligned}$$

Hence, for $n \geq N + 1$, we have that

$$\lambda(y_n) \geq \left(1 + \frac{1}{m}\right)\lambda(y_{n-1}) \geq \ldots \geq \left(1 + \frac{1}{m}\right)^{n-N}\lambda(y_N) > \left(1 + \frac{1}{m}\right)^{n-N}\lambda(x_1).$$

Since $\lambda(x_1) > 0$, it follows that $\lim_{n\to\infty} \lambda(y_n) = \infty$; and hence $\lim_{n\to\infty} y_n = 0$. Hence $\lim_{n\to\infty}(x_1 + \sum_{k=1}^{n} y_k)$ exits in $\mathcal{R}$. Let $x = \lim_{n\to\infty}(x_1 + \sum_{k=1}^{n} y_k)$. Then $x \approx x_1$ and hence $X_0 + x \approx X_0 + x_1$. Moreover, since $X_0 + x_1 + \sum_{k=1}^{n} y_k \in [0,1]$ for all $n \geq 1$, it follows that

$$X_0 + x = X_0 + \lim_{n\to\infty}\left(x_1 + \sum_{k=1}^{n} y_k\right) = \lim_{n\to\infty}\left(X_0 + x_1 + \sum_{k=1}^{n} y_k\right) \in [0,1].$$

Since $\lambda(f(X_0 + x_1 + \sum_{k=1}^{n} y_k)) > \lambda(y_n)$ and since $\lim_{n\to\infty} y_n = 0$, it follows that $\lim_{n\to\infty} f\left(X_0 + x_1 + \sum_{k=1}^{n} y_k\right) = 0$. Thus,

$$\begin{aligned}
f(X_0 + x) &= f\left(X_0 + \lim_{n\to\infty}\left(x_1 + \sum_{k=1}^{n} y_k\right)\right) = f\left(\lim_{n\to\infty}\left(X_0 + x_1 + \sum_{k=1}^{n} y_k\right)\right) \\
&= \lim_{n\to\infty} f\left(X_0 + x_1 + \sum_{k=1}^{n} y_k\right) = 0.
\end{aligned}$$

Subcase I-2: $1 < l = m$. The search for an intermediate value here follows the same steps as in Subcase I-1 except that l is replaced by m in the first two iterations and the two equations (4.18) and (4.19) take the simpler form

$$\lambda\left(f^{(m)}(X_0 + x_1)\right) = \lambda\left(P^{(m)}(x_1)\right) = 0.$$

After the second iteration, we proceed exactly as in Subcase I-1.

Case II: $m = 1$. In this case, the (quasi-)multiplicity l of the $\mathcal{R}$-root x_1 of $P(x)$ is also equal to 1 since $1 \leq l \leq m = 1$. Hence the order of the leading polynomial agrees with the quasi-multiplicity of its $\mathcal{R}$-root (both equal to 1) from the first iteration on. Thus, the search for an intermediate value in this case is similar to that in Subcase I-2 (or Subcase I-1) except that, in this case, $N = 1$. □

Using Theorem 3.6, we obtain the following result.

Theorem 4.8: *Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$. Then f is infinitely often differentiable on $[a, b]$, and for any positive integer m, we have that $f^{(m)}$ is $\mathcal{R}$-analytic on $[a, b]$. Moreover, if f is given locally around $x_0 \in [a, b]$ by $f(x) = \sum_{n=0}^{\infty} a_n(x_0)(x - x_0)^n$, then $f^{(m)}$ is given by $f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1)\, a_n(x_0)(x - x_0)^{n-m}$. In particular, we have that $a_m(x_0) = f^{(m)}(x_0)/m!$ for all $m = 0, 1, 2, \ldots$.*

Finally, we close this paper with the following conjecture.

Conjecture 4.9: *(Extreme Values Theorem) Let $a < b$ in $\mathcal{R}$ be given, and let $f : [a, b] \to \mathcal{R}$ be $\mathcal{R}$-analytic on $[a, b]$. Then f assumes a maximum and a minimum on $[a, b]$.*

Remark: Ongoing research aims at proving the Extreme Values Theorem stated above. Once this conjecture has been proved, Rolle's Theorem and the Mean Value Theorem follow readily.

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