

# ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS ON THE LEVI-CIVITA FIELD

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**Abstract:** The solution of an ordinary differential equation with constant coefficients on the Levi-Civita field is investigated. Contrary to the real case, we first show that even when the order of the equation is finite, it has infinitely many linearly independent solutions; moreover, any corresponding initial value problem has infinitely many solutions.

The problems mentioned above are due to the disconnectedness of the field in the order topology and to the existence of nontrivial field automorphisms, and they can be solved by restricting the solutions to analytic functions. We show that an  $m$ th order ordinary differential equation with constant coefficients has exactly  $m$  linearly independent analytic solutions and that a given initial value problem has a unique analytic solution.

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## 1. INTRODUCTION

In this paper, we discuss the solutions of ordinary differential equations with constant coefficients on the Levi-Civita field  $\mathcal{R}$  [5, 6]. We

start with a review of some basic and useful terminology and refer the reader to [1, 2, 12, 13, 14] for a more detailed study of the field.

**Definition 1.1** (The set  $\mathcal{R}$ ). We define

$$\mathcal{R} = \{f : \mathbb{Q} \rightarrow \mathbb{R} \mid \{x \mid f(x) \neq 0\} \text{ is left-finite}\}.$$

So the elements of  $\mathcal{R}$  are those real valued functions on  $\mathbb{Q}$  that are non-zero only on a left-finite set, i.e. below every rational number  $q$  there are only finitely many points where the given functions do not vanish.

We denote elements of  $\mathcal{R}$  by  $x$ ,  $y$ , etc. and identify their values at  $q \in \mathbb{Q}$  with brackets like  $x[q]$ . For the further discussion, it is convenient to introduce the following terminology:

**Definition 1.2** ( $\text{supp}$ ,  $\lambda$ ,  $\sim$ ,  $\approx$ ,  $=_r$ ). For  $x \in \mathcal{R}$ , we define

$\text{supp}(x) = \{q \in \mathbb{Q} \mid x[q] \neq 0\}$  and call it the support of  $x$ ;

$\lambda(x) = \min(\text{supp}(x))$  for  $x \neq 0$  (which exists because of left-finiteness) and  $\lambda(0) = +\infty$ .

Comparing two elements  $x$  and  $y$  in  $\mathcal{R}$ , we say

$x \sim y$  if  $\lambda(x) = \lambda(y)$ ;

$x \approx y$  if  $\lambda(x) = \lambda(y)$  and  $x[\lambda(x)] = y[\lambda(y)]$ ;

$x =_r y$  if  $x[q] = y[q]$  for all  $q \leq r$ .

At this point, these definitions may feel somewhat arbitrary; but after having introduced the concept of ordering on  $\mathcal{R}$ , we will see that  $\lambda$  describes "orders of infinite largeness or smallness", the relation " $\approx$ " corresponds to agreement up to infinitely small relative error, while " $\sim$ " corresponds to agreement of order of magnitude.

**Definition 1.3** (Addition and Multiplication on  $\mathcal{R}$ ). We define addition on  $\mathcal{R}$  componentwise:  $(x + y)[q] = x[q] + y[q]$ . Multiplication is defined as follows: For  $q \in \mathbb{Q}$  we set

$$(1.1) \quad (x \cdot y)[q] = \sum_{\substack{q_x, q_y \in \mathbb{Q}, \\ q_x + q_y = q}} x[q_x] \cdot y[q_y].$$

Since elements of  $\mathcal{R}$  have left-finite supports, only finitely many terms contribute to the sum in Equation (1.1). It turns out that the operations  $+$  and  $\cdot$  make  $(\mathcal{R}, +, \cdot)$  into a field, in which we can isomorphically embed the reals as a subfield via the map  $\Pi : \mathbb{R} \rightarrow \mathcal{R}$  defined by

$$(1.2) \quad \Pi(x)[q] = \begin{cases} x & \text{if } q = 0 \\ 0 & \text{else} \end{cases}.$$

**Definition 1.4** (Ordering in  $\mathcal{R}$ ). Let  $x, y$  be distinct elements of  $\mathcal{R}$ . We say  $x > y$  if  $(x - y)[\lambda(x - y)] > 0$  and we say  $x < y$  if  $y > x$ .

With this definition of the order relation,  $\mathcal{R}$  is a totally ordered field. Moreover, the embedding  $\Pi$  in (1.2) is compatible with the order. Besides the usual order relations, some other notations are convenient:

**Definition 1.5.** ( $\ll, \gg$ ) Let  $a, b$  be non-negative. We say  $a$  is infinitely smaller than  $b$  (and write  $a \ll b$ ) if  $n \cdot a < b$  for all natural  $n$ ; we say  $a$  is infinitely larger than  $b$  (and write  $a \gg b$ ) if  $b \ll a$ . If  $a \ll 1$ , we say  $a$  is infinitely small; if  $1 \ll a$ , we say  $a$  is infinitely large. Infinitely small numbers are also called infinitesimals or differentials. Infinitely large numbers are also called infinite. Non-negative numbers that are neither infinitely small nor infinitely large are also called finite.

**Definition 1.6** (The Number  $d$ ). Let  $d$  be the element of  $\mathcal{R}$  given by  $d[1] = 1$  and  $d[q] = 0$  for  $q \neq 1$ .

It is easy to check that  $0 < d^q \ll 1$  if and only if  $q > 0$  and  $d^q \gg 1$  if and only if  $q < 0$ . It follows that, altogether, the Levi-Civita  $\mathcal{R}$  is a totally ordered non-Archimedean field extension of the real numbers.

From general properties of formal power series fields [7, 10], it follows that  $\mathcal{R}$  is algebraically closed (every polynomial of odd degree over  $\mathcal{R}$  has at least one root in  $\mathcal{R}$ ). For a general overview of the algebraic properties of formal power series fields, we refer to the comprehensive overview by Ribenboim [11], and for an overview of the related valuation theory the book by Krull [3]. A thorough and complete treatment of ordered structures can also be found in [9].

Besides being the smallest totally ordered non-Archimedean field extension of the real numbers that is both complete in the order topology and algebraically closed [1], the Levi-Civita field  $\mathcal{R}$  is of particular interest because of its practical usefulness. Since the supports of the elements of  $\mathcal{R}$  are left-finite, it is possible to represent these numbers on a computer [1]. Having infinitely small numbers, the errors in classical numerical methods can be made infinitely small and hence irrelevant in all practical applications. One such application is the computation of derivatives of real functions representable on a computer [13], where both the accuracy of formula manipulators and the speed of classical numerical methods are achieved.

## 2. INFINITE DIMENSIONALITY OF THE SOLUTION SPACE OF $y' = 0$

The following theorem shows that even the simplest differential equation  $y' = 0$  over  $\mathcal{R}$  has infinitely many linearly independent solutions on  $[-1, 1]$ .

**Theorem 2.1.** *The solution space of the differential equation  $y' = 0$  on  $[-1, 1]$  is infinite dimensional.*

*Proof.* For each  $n \in \mathbb{N}$ , let  $g_n : [-1, 1] \rightarrow \mathcal{R}$  be given by  $g_n(x)[q] = x[q/(n+1)]$ . Then we show that, for all  $n \in \mathbb{N}$ ,  $g_n$  is differentiable on  $[-1, 1]$  with  $g'_n(x) = 0$  for all  $x \in [-1, 1]$ . So let  $n \in \mathbb{N}$  be given. We first observe that  $g_n(x+y) = g_n(x) + g_n(y)$  for all  $x, y \in [-1, 1]$ . Now let  $x \in [-1, 1]$  and  $\epsilon > 0$  in  $\mathcal{R}$  be given. Let  $\delta = \min\{\epsilon^2, d\}$ , and let  $y \in [-1, 1]$  be such that  $0 < |y-x| < \delta$ . Then

$$\left| \frac{g_n(y) - g_n(x)}{y-x} \right| = \left| \frac{g_n(y-x)}{y-x} \right| \sim |y-x|^n \text{ since } g(y-x) \sim (y-x)^{n+1}.$$

Since  $|y-x| < \min\{\epsilon^2, d\}$ , we obtain that  $|y-x|^n \ll \epsilon$ . Hence

$$\left| \frac{g_n(y) - g_n(x)}{y-x} \right| < \epsilon \text{ for all } y \in [-1, 1] \text{ satisfying } 0 < |y-x| < \delta.$$

It follows that  $g_n$  is differentiable at  $x$ , with  $g'_n(x) = 0$ . This is true for all  $x \in [-1, 1]$  and for all  $n \in \mathbb{N}$ . Hence  $g_n$  is a solution of the differential equation  $y' = 0$  on  $[-1, 1]$  for all  $n \in \mathbb{N}$ . Next we show that the set  $S = \{g_n : n \in \mathbb{N}\}$  is linearly independent on  $[-1, 1]$ . So let  $j \in \mathbb{N}$  and let  $n_1 < n_2 < \dots < n_j$  in  $\mathbb{N}$  be given. We show that  $g_{n_1}, g_{n_2}, \dots, g_{n_j}$  are linearly independent on  $[-1, 1]$ . So suppose that  $c_1 g_{n_1} + c_2 g_{n_2} + \dots + c_j g_{n_j} = 0$  for some  $c_1, c_2, \dots, c_j$  in  $\mathcal{R}$ ; we show that  $c_1 = c_2 = \dots = c_j = 0$ . Since  $c_1 g_{n_1} + c_2 g_{n_2} + \dots + c_j g_{n_j} = 0$ , we obtain that  $c_1 g_{n_1}(d) + c_2 g_{n_2}(d) + \dots + c_j g_{n_j}(d) = 0$ . Hence  $c_1 d^{n_1} + c_2 d^{n_2} + \dots + c_j d^{n_j} = 0$ ; from which we infer that  $c_1 = c_2 = \dots = c_j = 0$ .  $\square$

**Remark 2.2.** For each  $n \in \mathbb{N}$ , it is easy to check that the mapping  $g_n$  in the proof of Theorem 2.1 is an order preserving field automorphism on  $\mathcal{R}$ ; this is a special property of non-Archimedean structures since it is well-known that the only order preserving field automorphism on  $\mathbb{R}$  is the identity map [12].

**Remark 2.3.** Since for each  $n$ , the function  $g_n$  in the proof of Theorem 2.1 satisfies the initial condition  $g_n(0) = 0$ , we obtain that contrary to the real case, the initial value problem ( $y' = 0$ ;  $y(0) = 0$ ) has infinitely many solutions on  $[-1, 1]$ .

**Corollary 2.4.** *Let  $m \in \mathbb{N}$  and  $a_1, \dots, a_m \in \mathcal{R}$  be given. Then the initial value problem*

$$(2.1) \quad y^{(m)} + a_1 y^{(m-1)} + \dots + a_m y = 0; \quad y(0) = 0, y'(0) = 0, \dots, y^{(m-1)}(0) = 0$$

*has infinitely many solutions on  $[-1, 1]$ .*

In Section 4, we restrict ourselves to solutions that are given locally by power series, which we will refer to as analytic solutions, and show that then the initial value problem in Equation (2.1) has the unique solution that is identically null around 0. Consequently, we show that the differential equation  $y^{(m)} + a_1y^{(m-1)} + \dots + a_m y = 0$  has exactly  $m$  linearly independent analytic solutions around 0. But we first review some properties of power series and we refer the interested reader to [12, 14] for more details.

### 3. REVIEW OF POWER SERIES

Power series on the Levi-Civita field  $\mathcal{R}$  and its complex counterpart  $\mathcal{C}$  have been studied in details in [12, 14]. Previous work on power series on  $\mathcal{R}$  and  $\mathcal{C}$  had been mostly restricted to power series with real or complex coefficients. In [5, 6, 8, 4], they could be studied for infinitely small arguments only, while in [1], using the newly introduced weak topology, also finite arguments were possible. In [12, 14], we study the general case when the coefficients are Levi-Civita numbers.

Convergence criteria for power series on  $\mathcal{R}$  and  $\mathcal{C}$  are derived in [12, 14], which allow us to define a radius of convergence  $\eta$  such that the power series converges weakly for all points whose distance from the center is smaller than  $\eta$  by a finite amount and it converges strongly for all points whose distance from the center is infinitely smaller than  $\eta$ . Then it is shown that power series on  $\mathcal{R}$  and  $\mathcal{C}$  behave similarly to real and complex power series. Specifically, within their radius of convergence, power series are infinitely often differentiable and the derivatives to any order are obtained by differentiating the power series term by term; moreover, power series can be re-expanded around any point in their domain of convergence. Finally, locally analytic functions are studied and we show that they satisfy all the common theorems of real calculus on a closed interval of  $\mathcal{R}$ , like the intermediate value theorem, the maximum theorem and the mean value theorem.

Of particular interest here is the following result which allows us to extend to  $\mathcal{R}$  and  $\mathcal{C}$  all real and complex functions that are given locally by power series [1, 12, 14].

**Theorem 3.1** (Power Series with Purely Real or Complex Coefficients). *Let  $\sum_{n=0}^{\infty} a_n X^n$  be a power series with purely real (or complex) coefficients and with classical radius of convergence equal to  $\eta$ . Let  $x \in \mathcal{R}$  (or  $\mathcal{C}$ ), and let  $A_n(x) = \sum_{i=0}^n a_i x^i \in \mathcal{R}$  (or  $\mathcal{C}$ ). Then, for  $|x| < \eta$  and  $|x| \not\approx \eta$ , the sequence  $(A_n(x))$  converges absolutely weakly. We define the limit to be the continuation of the power series to  $\mathcal{R}$  (or  $\mathcal{C}$ ).*

**Definition 3.2.** Using Theorem 3.1, the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}, \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \text{ and } \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

converge absolutely weakly in  $\mathcal{R}$  and  $\mathcal{C}$  for any  $x$ , at most finite in absolute value. We define these series to be  $\exp(x)$ ,  $\cos(x)$ ,  $\sin(x)$ ,  $\cosh(x)$  and  $\sinh(x)$ , respectively.

A detailed study of the transcendental functions introduced in Definition 3.2 can be found in [12]. In particular, it is easily shown that addition theorems similar to the real (or complex) ones hold for these functions.

#### 4. ANALYTIC SOLUTIONS OF ODE'S WITH CONSTANT COEFFICIENTS

In this section, we will consider  $m$ th order ordinary differential equations with constant coefficients. We show that if we restrict ourselves to analytic solutions, the solution space of such an  $m$ th order differential equation is  $m$ -dimensional, just like in the real case.

**4.1. First Order Differential Equations.** In this section we study the analytic solution of the differential equation  $y' + ay = 0$ , where  $a$  is a constant.

**Theorem 4.1.** *The initial value problem*

$$(4.1) \quad y' + ay = 0; \quad y(0) = 0$$

*has the unique analytic solution that is null around zero.*

*Proof.* Let  $y = \phi(x) = \sum_{n=0}^{\infty} \alpha_n x^n$  be an analytic solution of the initial value problem (4.1). Then, for all  $n \geq 0$ , we have that  $\phi^{(n)}(0) = n! \alpha_n$ . From Equation (4.1), we obtain that

$$\alpha_0 = \phi(0) = 0;$$

also,  $\phi'(x) + a\phi(x) = 0$  for all  $x$ ; and hence

$$\alpha_1 = \phi'(0) = -a\phi(0) = 0.$$

Now suppose that  $\alpha_k = 0$ ; we show that  $\alpha_{k+1} = 0$ . Differentiating the equation  $\phi'(x) + a\phi(x) = 0$ ,  $k$  times, we obtain that  $\phi^{(k+1)}(x) + a\phi^{(k)}(x) = 0$  for all  $x$ ; and hence

$$\alpha_{k+1} = \frac{\phi^{(k+1)}(0)}{(k+1)!} = -a \frac{\phi^{(k)}(0)}{(k+1)!} = -a \frac{k!}{(k+1)!} \alpha_k = 0.$$

Using the principle of mathematical induction, we obtain that  $\alpha_n = 0$  for all  $n \geq 0$ . □

**Corollary 4.2.** *Let  $a \in \mathcal{R}$  be given. Then the general analytic solution of the equation  $y' + ay = 0$  around 0 is given by  $y = c \exp(-ax)$ , where  $c$  is an arbitrary constant.*

*Proof.* That  $y = c \exp(-ax)$  is a solution can be verified by direct substitution. Now let  $y = \phi(x)$  be any analytic solution of  $y' + ay = 0$ ; we show that  $\phi(x) = c_0 \exp(-ax)$  for some  $c_0 \in \mathcal{R}$ . Let  $c_0 = \phi(0)$ ; and let  $\psi(x) = \phi(x) - c_0 \exp(-ax)$ . Then  $\psi$  is an analytic solution of the initial value problem, Equation (4.1); and hence  $\psi$  is identically null around 0 by Theorem 4.1. Thus,  $\phi(x) = c_0 \exp(-ax)$ .  $\square$

**Remark 4.3.** By Theorem 3.1, the solution  $y = \exp(-ax)$  is defined on any interval  $(-\delta, \delta)$  around 0 such that  $|a|\delta$  is not infinitely large (i.e. at most finite.)

**4.2. Second Order Differential Equations.** In this section we study the analytic solution of the differential equation  $y'' + a_1y' + a_2y = 0$ , where  $a_1$  and  $a_2$  are constants.

**Theorem 4.4.** *The initial value problem*

$$(4.2) \quad y'' + a_1y' + a_2y = 0; \quad y(0) = 0, y'(0) = 0$$

*has the unique analytic solution that is identically null around zero.*

*Proof.* Let  $y = \phi(x) = \sum_{n=0}^{\infty} \alpha_n x^n$  be an analytic solution of the initial value problem in Equation (4.2). Then, for all  $n \geq 0$ , we have that  $\phi^{(n)}(0) = n! \alpha_n$ . It follows from Equation (4.2) that  $\alpha_0 = \phi(0) = 0$  and  $\alpha_1 = \phi'(0) = 0$ . Now suppose that  $\alpha_k = \alpha_{k+1} = 0$ ; we show that  $\alpha_{k+2} = 0$ . Differentiating the equation  $\phi''(x) + a_1\phi'(x) + a_2\phi(x) = 0$ ,  $k$  times, we obtain that  $\phi^{(k+2)}(x) + a_1\phi^{(k+1)}(x) + a_2\phi^{(k)}(x) = 0$  for all  $x$ ; and hence

$$\begin{aligned} \alpha_{k+2} &= \frac{\phi^{(k+2)}(0)}{(k+2)!} \\ &= -a_1 \frac{\phi^{(k+1)}(0)}{(k+2)!} - a_2 \frac{\phi^{(k)}(0)}{(k+2)!} \\ &= -a_1 \frac{(k+1)!}{(k+2)!} \alpha_{k+1} - a_2 \frac{k!}{(k+2)!} \alpha_k = 0. \end{aligned}$$

Again, using the principle of mathematical induction, we obtain that  $\alpha_n = 0$  for all  $n \geq 0$ .  $\square$

**Corollary 4.5.** *Let  $a_1, a_2 \in \mathcal{R}$  be given, and let  $\Delta = a_1^2 - 4a_2$ . Then the general analytic solution of the equation  $y'' + a_1y' + a_2y = 0$  is given by*

- $y = c_1 \exp(r_1 x) + c_2 \exp(r_2 x)$  if  $\Delta > 0$ , where  $r_1$  and  $r_2$  are the distinct roots of the characteristic equation  $r^2 + a_1 r + a_2 = 0$  in  $\mathcal{R}$ .
- $y = c_1 \exp(rx) + c_2 x \exp(rx)$  if  $\Delta = 0$ , where  $r$  is the double root of  $r^2 + a_1 r + a_2 = 0$  in  $\mathcal{R}$ .
- $y = c_1 \exp(\lambda x) \cos(\mu x) + c_2 \exp(\lambda x) \sin(\mu x)$  if  $\Delta < 0$ , where  $\lambda \pm i\mu$  are the roots of  $r^2 + a_1 r + a_2 = 0$  in  $\mathcal{C}$ .

*Proof.* The proof is done by direct substitution and making use of Theorem 4.4, just like we used Theorem 4.1 in the proof of Corollary 4.2. We note that, given some initial conditions  $y(0) = y_0$  and  $y'(0) = y'_0$ , then there is a unique choice of the constants  $c_1$  and  $c_2$  that would satisfy those conditions.  $\square$

**Remark 4.6.** Since the arguments of the functions  $\exp$ ,  $\cos$  and  $\sin$  have to be at most finite in absolute value, the domain of definition of the solutions given in Corollary 4.5 depends on the size of the constants  $a_1$  and  $a_2$ .

**4.3. Higher Order Differential Equations.** The results in Section 4.2 are readily generalized to any order  $m$ , just like in the real case. That the initial value problem

$$y^{(m)} + a_1 y^{(m-1)} + \cdots + a_m y = 0; \quad y(0) = 0, y'(0) = 0, \dots, y^{(m-1)}(0) = 0$$

has the unique analytic solution that is identically null around 0 can be proved in the same way as Theorem 4.1 and Theorem 4.4, by successively differentiating the differential equation and using induction. Then the general solution of the differential equation  $y^{(m)} + a_1 y^{(m-1)} + \cdots + a_m y = 0$  is a linear combination of  $m$  linearly independent solutions whose forms depend on the nature of the roots of the characteristic equation  $r^m + a_1 r^{m-1} + \cdots + a_{m-1} r + a_m = 0$ , just as in the real case. Again, the size of the domain of definition of the solution depends on the size of the constants  $a_1, a_2, \dots, a_m$ , by Theorem 3.1.

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