A NOTE ON A THEOREM OF CADORET AND TAMAGAWA

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ABSTRACT. We prove Cadoret and Tamagawa’s open image theorem for curves defined over number fields using their arguments and the machinery of Ellenberg-Hall-Kowalski employed in their paper on expander graphs.

Let $\ell$ be a prime. A compact lie group $G$ is called strictly rationally perfect if for each open subgroup $U \subseteq G$, its abelianization $U^{ab}$ is finite.

Let $k$ be a finitely generated field over $\mathbb{Q}$ and $\bar{k}$ be its algebraic closure. Let $X$ be a smooth, geometrically connected, and separated curve over $k$. Let $\rho : \Pi_1(X/k) \to \text{GL}_m(\mathbb{Z}_\ell)$ be an $\ell$-adic representation. Set $G := \rho(\Pi_1(X/k))$ and $\bar{G} := \rho(\Pi_1(X/\bar{k}))$.

Given a closed point $x \in X$, we denote the induced Galois representation of the absolute Galois group $G_{k(x)}$ of the residue field $k(x)$ by $\rho_x : G_{k(x)} \to G$.

Set $G_x := \text{Im}(\rho_x) \subseteq G$.

In [1], Cadoret and Tamagawa recently showed that if $\bar{G}$ is strictly rationally perfect, then the images of Galois representations $\rho_x$ induced by the $\ell$-adic representation $\rho$ are generically open in $G$; that is for every $d$, the set

$$\{x \in X(K) \mid [K : k] \leq d \text{ and } G_x \text{ is not open in } G\}$$

is finite.

At around the same time, in [2], Ellenberg-Hall-Kowalski (EHK) proved that $\mathbb{C}$-gonality of curves $X_i$ in a tower of covers $X_i \to X$, where $X$ is a smooth curve defined over a number field, tends to $\infty$ as $i \to \infty$ if the associated family of Cayley-Schreier graphs satisfies certain expansion property. As applications of this general result, they proved several arithmetic statements in the spirit similar to the one of Cadoret and Tamagawa. They also state in the first draft of their paper that it seems likely to apply their arguments in the setting of Cadoret-Tamagawa.

Our goal in this short note is to prove the theorem mentioned above in case that $k$ is a number field using the ideas of EHK and Cadoret-Tamagawa. The proof is elementary modulo the result of EHK on gonality and the recent expansion result of Salehi-Golsefidy. We should also emphasize that all the arguments below are contained in the papers of Cadoret-Tamagawa and EHK; they are not new.

While the short proof in this note works only when $k$ is a number field, it is valid when $\bar{G}$ has a perfect connected component (as opposed to $\bar{G}$ being strictly rationally perfect). Note that if $\bar{G}$ is strictly rationally perfect then the abelianization of its connected component is a finite connected group, which implies that the abelianization is trivial and that $\bar{G}$ has a perfect connected component.

Below, $k$ denotes a number field and $X$ denotes a smooth, geometrically connected, separated curve defined over $k$. Let $X^{k, \leq d}$ denote the set of all closed points $x \in X(K)$ with $[K : k] \leq d$.

**Theorem 1.** Let the notation be as above. Assume that $\bar{G} \subseteq \text{GL}_m(\mathbb{Z}_\ell)$ is the Zariski closure of the group generated by a finite, symmetric subset $S \subseteq \text{GL}_m(\mathbb{Z}[1/q])$, where $q \in \mathbb{Z}$ with $(\ell, q) = 1$. Assume also that the connected component of $\bar{G}$ is perfect. Let $C^{\leq d}$ be the set of all the closed points $x \in X^{k, \leq d}$ such that $G_x$ is not open in $G$. Then, for any given $d$, the set $C^{\leq d}$ is finite. Moreover, there exists an integer $B_{\rho, d}$ such that

$$[G : G_x] \leq B_{\rho, d} \quad \text{for all } x \in X^{k, \leq d} - C^{\leq d}.$$

2010 Mathematics Subject Classification. 14D10, 11F80.
1. The Proof

For a field \( F \supseteq k \), let \( \gamma_F(X) \) denote \( F \)-gonality of \( X \); that is the smallest possible degree of a dominant rational map \( X \to \mathbb{P}^1 \) defined over \( F \). We denote its \( k \)-gonality by \( \gamma(X) \). Note that if \( F \) is algebraically closed and \( E/F \) is a field extension, then \( \gamma_F(X) = \gamma_E(X) \), [4, Proposition A.1]. Therefore, since \( k \) is a number field, \( \gamma(X) = \gamma_C(X) \).

For the rest of the note, let \( \rho, G, \bar{S} \) be as above. Assume that \( \bar{G} \) has a perfect connected component.

Given a positive integer \( n \), let 
\[
\pi_n : \text{GL}_m(\mathbb{Z}_\ell) \to \text{GL}_m(\mathbb{Z}_\ell/\ell^n)
\]
be the natural projection. Let \( G_n := \pi_n(G) \) and \( G(n) \subseteq G \) be the normal subgroup fitting in the following short exact sequence
\[
1 \longrightarrow G(n) \longrightarrow G \overset{\pi_n}{\longrightarrow} G_n \longrightarrow 1.
\]
The groups \( G(n) \) and \( G_n \) are defined likewise. Let \( \tilde{S}_n := \pi_n(\bar{S}) \). We need the following expansion result.

**Theorem 2.** [5, Theorem 1] Let the notation be as above. Assume that the connected component of \( G \) is perfect. Then, the family \( \{C(G(n), \tilde{S}_n)\}_n \) of Cayley-Schreier graphs is an expander family.

Given a subgroup \( B \subseteq G \), let \( \Phi(B) \) denote its Frattini subgroup– the intersection of all of its maximal open subgroups. For a positive integer \( n \), let 
\[
H_n(G) = \{U \subseteq G \text{ open} \mid G(n-1) \not\subseteq U \text{ and } \Phi(G(n-1)) \subseteq U\}.
\]
Note that \( H_n(G) \) is finite for all \( n \in \mathbb{N} \), see [1, Lemma 3.2 (1)] and that we have
\[
G = G(0) \supseteq G(1) \supseteq G(2) \supseteq \ldots \supseteq G(n-1) \supseteq G(n)\).
\]
Moreover, \( \Phi(G(n-1)) = G(n) \) for \( n \gg 1 \), [1, Lemma 3.1(3)]. Here is a lemma, which will be used later.

**Lemma 3.** Let \( B \) be a closed subgroup of \( G \) and \( n \in \mathbb{N} \). If \( B \not\subseteq U \) for any \( U \in H_n(G) \), then \( G(n-1) \subseteq B \).

**Proof.** Let \( B \) be a closed subgroup of \( G \) that does not contain \( G(n-1) \). Consider the open subgroup 
\[
U := B \Phi(G(n-1)).
\]
We will show that \( U \) is in \( H_n(G) \). Otherwise, \( U \) contains \( G(n-1) \); i.e. \( G(n-1) = B \Phi(G(n-1)) \cap G(n-1) = (B \cap G(n-1)) \Phi(G(n-1)) \). By Frattini property, \( B \cap G(n-1) = G(n-1) \). A contradiction. \( \square \)

Let \( I = \{(n,U) \mid U \in H_n(G)\} \) be the index set. After ordering the elements of \( H_n(G) \), we can consider \( I \) as a well-ordered set in the obvious way. Given \( i = (n,U) \in I \), let \( N_i = G/U \cap G \). Note that for open subgroups \( U \) with sufficiently small index, there exists a unique \( n \) such that \( U \in H_n(G) \) (because for \( n \gg 1 \), \( \Phi(G(n-1)) = G(n) \)).

**Proposition 4.** The family of Cayley-Schreier graphs 
\[
C(N_i, \tilde{S}) \text{ for } i \in I
\]
is an expander family.

**Proof.** Let \( \lambda_1(\bar{G}_n, \bar{S}_n) \) denote the first non-zero eigenvalue of the combinatorial laplacian. By theorem 2, there exists a constant \( c > 0 \) such that
\[
\lambda_1(\bar{G}_n, \bar{S}_n) \geq c
\]
for all positive integers \( n \). Since \( G(n) \subseteq U \) for \( i = (n,U) \) with \( n \gg 1 \), there exists a graph covering
\[
C(\bar{G}_n, \bar{S}_n) \to C(N_i, \tilde{S}).
\]
As the pull-back of a eigenfunction is again an eigenfunction with the same eigenvalue, this implies that
\[
\lambda_1(N_i, \tilde{S}) \geq \lambda_1(\bar{G}_n, \bar{S}_n) \geq c
\]
for all but finitely many \( i \in I \). This completes the proof. \( \square \)
Let $i = (n, U) \in I$. Then, the finite index subgroup $\rho^{-1}(U) \subseteq \Pi_1(X/k)$ corresponds to an étale cover, which we denote by

$$\varphi_i : X_i \to X$$

(that is, $\Pi_1(X_i/k) = \rho^{-1}(U)$). Notice that $\rho$ induces a bijection $\Pi_1(X/\bar{k})/\Pi_1(X_i/\bar{k}) \cong \tilde{G}/U \cap \tilde{G} = N_i$ so that corresponding Cayley-Schreier graphs are isomorphic:

$$C(\Pi_1(X/\bar{k})/\Pi_1(X_i/\bar{k}), \rho^{-1}(\bar{S})) \cong C(N_i, \bar{S}).$$

Note also that, because the geometric fundamental group $\Pi_1(X/\bar{k})$ is the profinite completion of the topological fundamental group $\Pi_1(X(\mathbb{C}))$, we have a natural bijection

$$\Pi_1(X(\mathbb{C}))/\Pi_1(X_i(\mathbb{C})) \cong \Pi_1(X/\bar{k})/\Pi_1(X_i/\bar{k}).$$

**Theorem 5.** [2, Theorem 8] Let $X$ be a smooth, connected, projective curve over $\mathbb{C}$. Let $\{X_i\}_{i \in I}$ be an infinite family of étale covers of $X$. Let $S \subseteq \Pi_1(X(\mathbb{C}))$ be a fixed finite subset generating the topological fundamental group $\Pi_1(X(\mathbb{C}))$.

Assume that the family of Cayley-Schreier graphs $C(N_i, S)$ is an expander family, where $N_i = \Pi_1(X(\mathbb{C}))/\Pi_1(X_i(\mathbb{C}))$. Then,

$$\gamma_C(X_i) \to \infty \text{ as } i \to \infty.$$

**Theorem 6.** [3] Let $k$ be a finitely generated extension of $\mathbb{Q}$. And, let $Y$ be a smooth, proper, geometrically connected curve over $k$ with $k$-gonality $\gamma_k(Y)$. Then, for each fixed $d \in \mathbb{N}$, the set $Y^k_{C, \leq d}$ is finite whenever $\gamma_k(Y) \geq 2d + 1$.

Note that all the arguments in the following proof is due to Cadoret-Tamagawa, and contained in their paper [1].

**Proof of Theorem 1.** Let $d \geq 1$ be fixed and let the notation be as above. By Theorem 5, $\gamma(X_i) \to \infty$ as $i \to \infty$. Therefore, there exists an integer $N = N_{p, d}$ such that for all $i \geq N$, $\gamma_k(X_i) \geq \gamma(X_i) \geq 2d+1$. Let

$$X_N = \bigcup_{i = (n, U) \in H_N(G)} X_i.$$

By Theorem 6, the set $X^k_{N, \leq d}$ is finite, and so is its image $\varphi(X^k_{N, \leq d})$ under the étale cover $\varphi : X_N \to X$. On the other hand, if $x \in X^k_{N, \leq d} - \varphi(X^k_{N, \leq d})$ then $G_x \not\subseteq U$ for any $U \in H_N(G)$. Then, by Lemma 3, $G(N - 1) \subseteq G_x$ and so $G_x$ is open. Hence, $C^{\leq d} \subseteq \varphi(X^k_{N, \leq d})$ and the set $C^{\leq d}$ is finite.

Finally, if $x \in \varphi(X^k_{N, \leq d}) - C^{\leq d}$, then $G_x$ is open. So, let

$$B_{p, d} := \max \left\{ [G : G(N - 1)], [G : G_x] \mid x \in \varphi(X^k_{N, \leq d}) - C^{\leq d} \right\}.$$

By the discussion above, we have

$$[G : G_x] \leq B_{p, d} \text{ for all } x \in X^k_{N, \leq d} - C^{\leq d}.$$