Problem 1 (12 points): Given that the implicit relations
\[ 2x + y - 3z - 2u = 0 \text{ and } x + 2y + z + u = 0 \]
define \( x \) and \( u \) as differentiable functions of \( y \) and \( z \). Find \((\partial x/\partial y)_z\).

If we let \( F(x, y, z, u) = 2x + y - 3z - 2u \) and \( G(x, y, z, u) = x + 2y + z + u \), then
\[
\left( \frac{\partial x}{\partial y} \right)_z = -\frac{\frac{\partial(F,G)}{\partial(y,u)}}{\frac{\partial(F,G)}{\partial(x,u)}} = -\frac{\begin{vmatrix} F_y & F_u \\ G_y & G_u \end{vmatrix}}{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}} = -\frac{5}{4}.
\]

Problem 2 (18 points): Let \( F(x, y, z) = x^2 + y^2 + z^2 - 9 \).

a) Find the directional derivative of \( F \) at the point \( Q(2, 1, 2) \) in the direction of the vector \( \vec{V} = \hat{i} - 2\hat{j} + 2\hat{k} \).

The directional derivative of \( F \) at the point \( Q(2, 1, 2) \) in the direction of the vector \( \vec{V} = \hat{i} - 2\hat{j} + 2\hat{k} \) is given by \( D_{\vec{V}}F(Q) = \nabla F(Q) \cdot (\vec{V}/|\vec{V}|) \), where
\[
\nabla F(Q) = \left( 2x\hat{i} + 2y\hat{j} + 2z\hat{k} \right)(Q) = 4\hat{i} + 2\hat{j} + 4\hat{k}; \text{ and } |\vec{V}| = \sqrt{1^2 + (-2)^2 + 2^2} = 3.
\]
Hence
\[
D_{\vec{V}}F(Q) = (4\hat{i} + 2\hat{j} + 4\hat{k}) \cdot \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3} = \frac{1}{3}(4\hat{i} + 2\hat{j} + 4\hat{k}) \cdot (\hat{i} - 2\hat{j} + 2\hat{k}) = \frac{1}{3}(4 - 8 + 8) = \frac{8}{3}.
\]

b) Find the equation of the plane \((P)\) tangent to the sphere \((S): x^2 + y^2 + z^2 - 9 = 0\) at the point \(Q\) of part a). (Note that \(Q \in (S)\)).

Since \( \nabla F(Q) = 4\hat{i} + 2\hat{j} + 4\hat{k} \) is normal to \((P)\) at \(Q(2, 1, 2)\), the equation of \((P)\) is given by:
\[
4(x - 2) + 2(y - 1) + 4(z - 2) = 0; \text{ or } 2x + y + 2z = 9.
\]

Problem 3 (16 points): Given the differential equation
\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0, \tag{1}
\]
use the new variables: $u$ (dependent variable), $z = x + y$ (independent variable) and $w = x - y$ (independent variable) to obtain a new differential equation involving $u$, $z$, $w$ and the partial derivatives of $u$ with respect to $z$ and $w$. Assume all the partial derivatives involved are continuous.

Using the Chain Rule, we have that

$$
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial u}{\partial w} \frac{\partial w}{\partial x}, \text{ and } \frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial u}{\partial w} \frac{\partial w}{\partial y} = \frac{\partial u}{\partial z} - \frac{\partial u}{\partial w}.
$$

It follows that

$$
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial z} \right) \frac{\partial z}{\partial x} + \frac{\partial}{\partial w} \left( \frac{\partial u}{\partial w} \right) \frac{\partial w}{\partial x} = \frac{\partial^2 u}{\partial z^2} + 2 \frac{\partial^2 u}{\partial z \partial w} + \frac{\partial^2 u}{\partial w^2}, \quad (2)
$$

and

$$
\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial z} \right) \frac{\partial z}{\partial y} + \frac{\partial}{\partial w} \left( \frac{\partial u}{\partial w} \right) \frac{\partial w}{\partial y} = \frac{\partial^2 u}{\partial z^2} - 2 \frac{\partial^2 u}{\partial z \partial w} + \frac{\partial^2 u}{\partial w^2}; \quad (3)
$$

Substituting for $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ from Equations (2) and (3) into Equation (1), we get

$$
4 \frac{\partial^2 u}{\partial z \partial w} = 0; \text{ or } \frac{\partial^2 u}{\partial z \partial w} = 0.
$$

**Problem 4 (18 points):** Let

$$
\vec{V}(x, y, z) = z e^{x+y} \hat{i} + z(e^{x+y} - 1) \hat{j} + (e^{x+y} - y)\hat{k}.
$$

a) Show that $\nabla \times \vec{V} = \vec{0}$. 

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\[ \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z e^{x+y} & z(e^{x+y}-1) & (e^{x+y}-y) \end{vmatrix} = \hat{i} \left( \frac{\partial(e^{x+y} - y)}{\partial y} - \frac{\partial(z(e^{x+y} - 1))}{\partial z} \right) \\
+ \hat{j} \left( \frac{\partial(z e^{x+y})}{\partial z} - \frac{\partial(e^{x+y} - y)}{\partial x} \right) + \hat{k} \left( \frac{\partial(z(e^{x+y} - 1))}{\partial x} - \frac{\partial(z e^{x+y})}{\partial y} \right) \]

\[ = \hat{i}((e^{x+y} - 1) - (e^{x+y} - 1)) + \hat{j}(e^{x+y} - e^{x+y}) + \hat{k}(z e^{x+y} - z e^{x+y}) = \vec{0}. \]

b) Find all functions \( f(x, y, z) \) such that \( \nabla f = \vec{V} \).

A function \( f(x, y, z) \) satisfies \( \nabla f = \vec{V} \) if and only if

\[ \frac{\partial f}{\partial x} = V_1 = z e^{x+y}, \quad (4) \]
\[ \frac{\partial f}{\partial y} = V_2 = z(e^{x+y} - 1), \quad \text{and} \quad (5) \]
\[ \frac{\partial f}{\partial z} = V_3 = e^{x+y} - y. \quad (6) \]

Integrating Equation (4) with respect to \( x \), we get:

\[ f(x, y, z) = z e^{x+y} + g(y, z), \quad \text{with} \ g(y, z) \ \text{an arbitrary function of} \ y, z. \quad (7) \]

Differentiating \( f(x, y, z) \) with respect to \( y \) in Equation (7) and comparing with Equation (5), we obtain that

\[ \frac{\partial g}{\partial y} = -z, \ \text{from which we get} \ g(y, z) = -yz + h(z), \]

for some arbitrary function \( h(z) \). Thus,

\[ f(x, y, z) = z e^{x+y} - yz + h(z) \quad (8) \]

Differentiating \( f(x, y, z) \) with respect to \( z \) in Equation (8) and comparing with Equation (6), we obtain that

\[ h'(z) = 0, \ \text{from which we get} \ h(z) = C, \ \text{a constant}. \]

Thus, \( f(x, y, z) \) satisfies \( \nabla f = \vec{V} \) if and only if

\[ f(x, y, z) = z e^{x+y} - yz + C, \ \text{with} \ C \ \text{a constant}. \]
Problem 5 (16 points):

a) Let $R$ be the region in the $xy$-plane given by: $x^2 - y^2 \geq 0, 0 \leq x \leq 1$, and let $f(x, y)$ be continuous on $R$. Write
\[
\int_R \int f(x, y) dA
\]
as an iterated double integral.

Note that the inequalities defining $R$ can be rewritten as: $-x \leq y \leq x$, $0 \leq x \leq 1$. Hence $R$ is the triangular region with vertices $(0, 0), (1, 1), \text{ and } (1, -1)$. We integrate with respect to $y$ first: A typical vertical line enters $R$ at the line $y = -x$ and exits at the line $y = x$; hence the limits of integration for $y$ are from $-x$ to $x$. The limits of integration for $x$ are from 0 (the minimum value of $x$ in $R$) to 1 (the maximum value of $x$ in $R$). Thus,
\[
\int_R \int f(x, y) dA = \int_0^1 \int_{-x}^x f(x, y) dy dx.
\]

b) Let $R$ be the tetrahedron with vertices $(0, 0, 0), (1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$; and let $f(x, y, z)$ be continuous on $R$. Write
\[
\int_R \int f(x, y, z) dV
\]
as an iterated triple integral.

Because of the symmetry of $R$ with respect to $x, y$ and $z$, we can write the iterated integral in any of the six orders of integration. For example, (see the solution of Exercise 4(b) on page 235, in class),
\[
\int_R \int f(x, y, z) dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} f(x, y, z) dz dy dx.
\]