Hint for Exercise # 2 in Section 2.3

Let \( x_0 \in E \) be given; such an \( x_0 \) exists since \( E \) is nonempty. Since \( \sup E \not\in E \), it follows that \( x_0 < \sup E \). Let

\[
x_1 = \frac{x_0 + \sup E}{2}.
\]

Then \( x_0 < x_1 < \sup E \) (since \( E \) is bounded above and hence \( \sup E \) is finite!) Since \( x_1 < \sup E \), there exists \( y_1 \in E \) such that \( x_1 < y_1 \leq \sup E \) (by the Approximation Property for Suprema). Again, since \( \sup E \not\in E \), it follows that \( x_1 < y_1 < \sup E \). Thus,

\[
0 < \sup E - y_1 < \sup E - x_1 = \frac{\sup E - x_0}{2}.
\]

Repeat the step above to get \( y_2 \in E \) such that

\[
\frac{y_1 + \sup E}{2} < y_2 < \sup E.
\]

Show then that

\[
0 < \sup E - y_2 < \frac{\sup E - y_1}{2} < \frac{\sup E - x_0}{4}.
\]

\[\vdots\]

Use induction to construct a sequence \( \{y_n\} \) of elements in \( E \) such that:

\[
\frac{y_{n-1} + \sup E}{2} < y_n < \sup E.
\]

Show that

\[
y_n < y_{n+1} \text{ and } 0 < \sup E - y_n < \frac{\sup E - x_0}{2^n} \text{ for all } n \in \mathbb{N}.
\]

Say why then it follows that the resulting sequence \( \{y_n\} \) is strictly increasing and converges to \( \sup E \).

Fill in the necessary details of the arguments above to get full credit for this Exercise.