Review for Test 3

Coverage: Basis, dimension, orthogonal bases, linear transformations, least-square approximations, eigenvalues and eigenvectors, determinants and characteristic polynomial.

1. (a) Explain a procedure for finding a basis for the row space of a matrix $A$.
   (b) Find a basis for the row space of the matrix
   
   $$A = \begin{bmatrix}
   1 & 2 & 1 & -1 \\
   1 & 2 & 2 & -3 \\
   2 & 4 & 3 & -4 \\
   -3 & -6 & -5 & 7
   \end{bmatrix}.$$ 

2. (a) Explain a procedure for finding a basis for the column space of a matrix $A$.
   (b) Find a basis for the column space of the matrix
   
   $$A = \begin{bmatrix}
   1 & 2 & 1 & -1 \\
   1 & 2 & 2 & -3 \\
   2 & 4 & 3 & -4 \\
   -3 & -6 & -5 & 7
   \end{bmatrix}.$$ 

3. (a) Explain a procedure for finding a basis for the null space of a matrix $A$.
   (b) Find a basis for the null space of the matrix
   
   $$A = \begin{bmatrix}
   1 & 2 & 1 & -1 \\
   1 & 2 & 2 & -3 \\
   2 & 4 & 3 & -4 \\
   -3 & -6 & -5 & 7
   \end{bmatrix}.$$ 

   What is the dimension of $\mathcal{N}(A)$? Justify your answer!

4. Check all correct statements.
   - Two vectors in a 3-dimensional subspace are linearly independent.
   - Five vectors in a 3-dimensional subspace $W$ span $W$.
   - Three vectors in a 3-dimensional subspace are a basis of $W$.
   - 3 linearly independent vectors in a 3-dimensional subspace $W$ are a basis of $W$.
   - Three vectors spanning a 3-dimensional subspace $W$ are a basis of $W$.
   - Seven vectors in a 3-dimensional subspace are linearly dependent.

5. Use Gramm-Schmidt to find an orthogonal basis of $\text{Sp}(v_1, v_2, v_3)$ where
   
   $$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$ 
   $$v_2 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix},$$ 
   and 
   $$v_3 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$
6. (a) Complete the following definition.

A transformation $T : V \to W$ is linear if . . .

(b) Show that $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$
T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 - 2x_2 - x_3 \\ 3x_1 - 5x_2 + x_3 \end{bmatrix}
$$

is linear. Find a matrix $A$ such that $T(x) = Ax$ for all $x \in \mathbb{R}^3$.

7. (a) Find a formula for the rotation $R$ of $\mathbb{R}^2$ around the origin by angle $\pi/3$.

8. (a) Describe how to find the least-square solution of the equation $Ax = b$.

(b) Find the line that is the least-squares approximation to the data below.

<table>
<thead>
<tr>
<th>$t$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

9. Check all vectors that are eigenvectors of

$$
A = \begin{bmatrix}
-2 & -1 & 0 \\
0 & 1 & 1 \\
-2 & -2 & -1
\end{bmatrix}.
$$

\[ \circ \begin{bmatrix} 1 \\ 2 \\ -2 \\ 2 \end{bmatrix} \circ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \circ \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \circ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

10. Find the characteristic polynomial and the eigenvalues of

$$
A = \begin{bmatrix}
2 & 4 & 4 \\
0 & 1 & -1 \\
0 & 1 & 3
\end{bmatrix}.
$$

11. Find the maximal number of linearly independent eigenvectors of

$$
A = \begin{bmatrix}
2 & 1 & 3 \\
0 & 2 & 1 \\
0 & 0 & 1
\end{bmatrix}.
$$