Linear Dependence and Replacement Lemmas

**Linear Dependence Lemma.** Let $U$ and $W$ be subsets of a vector space $V$, such that $U$ is linearly independent. If $W \cup U$ is linearly dependent, then there exists a $w \in W$ such that $\text{Span}(U \cup (W - \{w\})) = \text{Span}(U \cup W)$.

**Proof:** Since $U \cup W$ is linearly dependent, there exists a non-trivial linear combination of its elements which is zero. Thus, there exist vectors $u_1, \ldots, u_k \in U, w_1, \ldots, w_m \in W$ and scalars $a_1, \ldots, a_k, b_1, \ldots, b_m \in \mathbb{F}$, such that

$$a_1 u_1 + \cdots + a_k u_k + b_1 w_1 + \cdots + b_m w_m = 0 \quad (1)$$

and not all $a_1, \ldots, a_k, b_1, \ldots, b_m$ are zero.

At least one of the numbers $b_1, \ldots, b_m$ is non-zero. For if that was not the case, (1) would give us $a_1 u_1 + \cdots + a_k u_k = 0$ with not all of the coefficients zero, contradicting the linear independence of $U$. Let $b_j$ be a non-zero coefficient. Then, solving (1) for $b_j w_j$, we get

$$b_j w_j = (-a_1) u_1 + \cdots + (-a_k) u_k + (-b_1) w_1 + \cdots + (-b_j) w_j + \cdots + (-b_m) w_m,$$

where $(-b_j) w_j$ indicates that $(-b_j) w_j$ is omitted. Since $b_j$ is non-zero, $b_j^{-1}$ exists, therefore we can solve the above equality for $w_j$ to get

$$w_j = (-a_1 b_j^{-1}) u_1 + \cdots + (-a_k b_j^{-1}) u_k + (b_1 b_j^{-1}) w_1 + \cdots + (b_j b_j^{-1}) w_j + \cdots + (b_m b_j^{-1}) w_m.$$

Therefore $w_j$ is in the span of $U \cup (W - \{w_j\}$). This implies that $U \cup W$ is a subset of $\text{Span}(U \cup (W - \{w_j\}))$ and therefore

$$\text{Span}(U \cup W) \subset \text{Span}(U \cup (W - \{w_j\}), \quad (2)$$

since the span of a set is the smallest subspace that contains that set. Since we also have $U \cup (W - \{w_j\}) \subset U \cup W$, we get

$$\text{Span}(U \cup (W - \{w_j\})) \subset \text{Span}(U \cup W),$$

which together with (2) implies that the two spans are equal. This finishes the proof of the lemma.

**Replacement Lemma.** Let $U$ and $W$ be subsets of a vector space $V$, such that $U$ has $m < \infty$ elements and is linearly independent, and such that $U \subset \text{Span}(W)$. Then there exists a subset $W_1$ of $W$ having $m$ elements, such that $\text{Span}(U \cup (W - W_1)) = \text{Span}(W)$. In particular, $W$ has to have at least $m$ elements.

**Proof:** By induction on $m \geq 0$. Clear for $m = 0$.

Suppose the statement is true when for $|U| < m$, $m > 0$ ($|U|$ denotes the cardinality of $U$). Assume now that $|U| = m$. Let $u \in U$ and let $U_1 = U - \{u\}$. Then $|U_1| = m - 1$. Since $U_1 \subset U \subset \text{Span}(W)$, $U_1$ and $W$ are satisfying the conditions of the lemma. Therefore, by the inductive hypothesis, there exists a subset $W'$ of $W$ having $m - 1$ elements, such that $\text{Span}(U_1 \cup (W - W')) = \text{Span}(W)$. Since $u \in U \subset \text{Span}(W) = \text{Span}(U_1 \cup (W - W'))$, the set $U \cup W = (U_1 \cup \{u\}) \cup (W - W')$ is linearly dependent. Therefore $U$ and $W - W'$ are satisfying the conditions of the Linear Dependence Lemma, which gives the existence of a vector $w \in W - W'$ such that $\text{Span}(U \cup ((W - W') - \{w\})) = \text{Span}(U \cup (W - W')) = \text{Span}(W)$. Clearly $W_1 = W' \cup \{w\}$ is the desired set.